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Names of Contributors are in Small Capitals.

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THE

# MATHEMATICAL MONTHLY.

## INTRODUCTORY NOTE.

The attempt to establish a Mathematical Journal is a step of too great importance to be taken without due deliberation, — without carefully considering the end to be attained, and the means to be employed in securing it. This end may be either the *advancement* of the science, or the *elevation of the standard*, of mathematical learning. Now it is not probable that a journal of a high scientific character, having the former end solely in view, could be sustained, for it would contain only one element of interest, and that one for the few professed mathematicians.

But a journal having the latter end in view, if successful in the highest sense, must necessarily to a greater or less degree involve the former; and the question arises, May not such a journal have a scope sufficiently comprehensive and elastic to embrace all grades of talent and attainment, and, therefore, corresponding elements of interest? If so, then it should embrace students in one extreme, and professed mathematicians in the other; which extremes neces-

sarily include all intermediate grades of teachers and laborers in this vast field.

Should it be a journal merely of problems and solutions? We think not. It should cover the *whole ground* sketched in the following circular note.

To the student, or younger mathematician, problems are indispensable to make him *sure* of, and *ground* him in, the theory, as well as exercise and develop his skill. But it is a serious question, whether it is not a waste of time and energy to put scores of able mathematicians upon the same problem, any one of whom has the knowledge and skill necessary for the complete solution. The problems should, therefore, usually be selected with reference to the learner, and so graduated as to suit different degrees of knowledge and skill even in the same branch.

Problems of the highest grade, especially if they are likely to lead the investigator into a comparatively new field, or develop methods or important practical results, may occasionally be published as *challenges;* but generally, we think it advisable to publish the solution of such problems at once, and if those particularly interested in the solution should be led to any new and curious developments, insert them afterwards.

The Journal should contain, for all learners, clear and concise notes upon all points of theory and application in all branches of the science; and these notes should come from able contributors, who can be plain without being weak,— who can unite simplicity of treatment with elegance of style.

It should contain "all scraps of mathematical writing too good to be lost," whether *elementary* or *profound,* whether original in *manner* or *matter,* whether complete in themselves or to be resumed at the convenience of the author, whether notices or reviews of matter old or new; in short, every thing fitly designated by "*notes and queries.*"

Besides, it should contain "carefully elaborated essays, chiefly valuable perhaps as promises of better things hereafter," as well as those of a higher character.

There is another large field in which the Journal will find its legitimate work,—one in which it can do the double duty of inducting students and younger mathematicians into the highest departments of the science, and of opening to the abler and more experienced an opportunity to contribute their share to the noble work of elevating the standard of mathematical learning in the country. All mathematicians know that there are many subjects in the higher departments of the science upon which little, if any thing, has as yet been written among us. Now, if they will take these subjects and develop them fully and systematically through the pages of the Journal, they may afterwards be issued in a separate form from the stereotype plates, at a very small cost. In such cases the right and benefit thereof shall vest in the author.

In this way we shall secure the coöperation of all: of students and younger mathematicians, for the range embraces them with their respective abilities and attainments, and therefore interests; of professed mathematicians, for, besides the large field specified above, neither their dignity nor scientific character can be affected by communicating notes which might not be of sufficient importance to warrant insertion in a Journal of high scientific pretensions.

In fine, then, the Journal will be to the professed mathematician a *recreation* and a *study*, while to the student it will be a *study* and an *example*.

Being convinced that a Journal of this character, in which all interests shall blend and coöperate, is needed, that it "will occupy ground unoccupied by other periodicals, and will be of great importance in advancing the intellectual character of our country;" and believing, that, if properly sustained by those who ought to have its

success most deeply at heart, it can be sustained in its financial department, we have taken the liberty during the last few weeks, as a preliminary inquiry, to send copies of the following note. The replies already received are of such a character as to fully warrant its issue in this more general manner. Every one, feeling any interest in this enterprise, is earnestly solicited to express his views fully, and state to what extent, if any, he will coöperate; so that his name may be included in the following list. As this introductory note will be substantially retained in the first number, the names of many of those who will give to the Journal its vitality and usefulness will thus be enrolled together: an array and diversity of talent that will make it a means of culture, which the friends of good learning in our country "will not willingly let die."

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## CIRCULAR NOTE.

NAUTICAL ALMANAC OFFICE, CAMBRIDGE, February 13, 1858.

DEAR SIR,— Allow me to call your attention to the following considerations: You are aware, that, while almost every science, as well as art, has its own appropriate journal, around which corresponding interests and tastes cluster, by which special research is encouraged, and through which all the valuable results are communicated to the world, the science of Mathematics is still without its own particular organ.

Now it seems to us that such a journal is needed; one that shall embrace, among its contributors, the best talent, in order that younger laborers in the same field may always have before them a high standard of excellence, and that it may be a fair index of the mathematical ability of the country. On the other hand, however, care should be taken not to graduate it, as a whole, too high above the average attainments of mathematical students; otherwise, only the few would be interested in it or benefited by it. It should therefore embrace in its pages solutions, demonstrations, and discussions in all branches of the science, as well as in all its various applications.

It should contain notes and queries, notices and reviews of all the principal mathematical works issued in this country, as well as in Europe.

In short, it should be the medium of all kinds of information pertaining to the science, to which a large proportion of our mathematical students have at present no ready access.

Such is, in brief, our idea of the character the journal should possess to insure to it the greatest usefulness and most permanent success.

This note, with the following queries, will be addressed to many of the most eminent mathematicians and educators in the various parts of the United States; and upon the several replies we shall base our future action.

Do you think there is a present need of a mathematical journal of any kind?

Do your views coincide with those here expressed as to its character? If not, be pleased to state your views.

Are you willing to assist in establishing and sustaining a journal by contributing to its pages?

Will you allow such use to be made of your reply to this note as may be proper to carry the proposed plan into effect?

A *decided* reply is respectfully solicited, whether favorable or otherwise.

With much esteem,

Yours truly,

J. D. Runkle.

---

To this note the following gentlemen have already replied:—

*Mr. A. Agassiz, Cambridge, Mass.
Prof. Stephen Alexander, College of New Jersey, Princeton.
*Prof. James L. Alverson, Genesee Coll., Lima, N. Y.
Prof. Theodore Appel, Franklin and Marshall Coll., Lancaster, Pa.
Prof. Samuel Alsop, West Chester, Penn.
Prof. John Aubier, St. John's Coll., Fordham, N. Y.
*Prof. Charles Avery, Hamilton Coll., Clinton, N. Y.
*Mr. R. S. Avery, U. S. Coast Survey, Washington, D. C.
*Dr. A. D. Bache, Sup't U. S. Coast Survey, Washington, D. C.
Prof. Mark Bailey, Franklin Coll., Franklin, Ind.
*Mr. F. W. Bardwell, Naut. Alm. Office, Cambridge, Mass.
*President F. A. P. Barnard, Univ. of Miss., Oxford.
Hon. Henry Barnard, Hartford, Conn.
Major J. G. Barnard, U. S. A., New York.
Prof. W. H. C. Bartlett, Military Academy, West Point.
*Prof. A. T. Bledsoe, University of Virginia.
*Prof. G. P. Bond, Harvard Coll. Obs., Cambridge, Mass.
Hon. G. S. Boutwell, Sec'y Board of Education, Boston, Mass.
Mr. J. Ingersoll Bowditch, Boston, Mass.
Prof. Francis Bowen, Harvard Coll., Cambridge, Mass.
*Mr. Isaac Bradford, Naut. Alm. Office, Cambridge, Mass.
Prof. John Brocklesby, Trinity Coll., Hartford, Conn.
Prof. P. P. Brown, Madison Univ., Hamilton, N. Y.
President John H. Bruner, Hiwasse Coll., Monroe Co., Tenn.
Dr. F. Brünnow, Director Obs., Ann Arbor, Mich.
Prof. John L. Campbell, Wabash Coll., Crawfordsville, Indiana.
Prof. David J. Capron, St. John's Coll., Annapolis, Md.
Prof. Alexis Caswell, Brown Univ., Providence, R. I.
*Prof. Wm. Chauvenet, Naval Academy, Annapolis, Md.

*Prof. J. B. Cherriman, University Coll., Toronto, Canada West.
Prof. A. E. Church, Military Academy, West Point.
Prof. James Clark, Georgetown Coll., D. C.
Prof. John E. Clark, Univ. of Michigan, Ann Arbor.
*Prof. A. W. Clarke, Washington Coll., Chestertown, Md.
*Prof. Geo. W. Coaklay, St. James' Coll., Washington Co., Md.
*Prof. J. H. C. Coffin, Naval Academy, Annapolis, Md.
Prof. James H. Coffin, LaFayette Coll., Easton, Pa.
Prof. J. H. Coit, St. James' Coll., Washington Co., Md.
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President Wm. Carey Crane, Semple-Broaddus Coll., Centre Hill, Miss.
President N. M. Crawford, Mercer Univ., Penfield, Georgia.
Prof. James D. Dana, Yale Coll., New Haven, Conn.
Prof. Charles Davies, Columbia Coll., New York.
Prof. S. S. Doak, Hiwassee Coll., Monroe Co. Tenn.
Prof. P. W. Dodson, Union Univ., Murfreesboro, Tenn.
Prof. F. B. Downes, Homer Academy, Homer, N. Y.
*Mr. John Downes, United States Coast Survey, Washington, D. C.
Prof. John T. Duffield, Coll. of New Jersey, Princeton.
*Mr. George Eastwood, Naut. Alm. Office, Cambridge, Mass.
Prof. C. W. Eliot, Harvard Coll., Cambridge, Mass.
*Mr. E. B. Elliott, Consulting Actuary, Boston.
*Prof. Ferdinand Engel, U. S. Coast Survey, Washington, D. C.
*Prof. H. L. Eustis, Lawrence Scientific School, Cambridge, Mass.
Prof. E. W. Evans, Marietta Coll., Ohio.
Hon. Edward Everett, Boston.
Prof. Benjamin S. Ewell, William & Mary's Coll., Williamsburg, Va.
Prof. J. H. Fairchild, Oberlin Coll., Ohio.
*Prof. W. Ferrel, Nashville, Tenn.
Prof. M. H. Fisk, Paducah Coll., Kentucky.
Mr. Oscar C. Fox, Prin. Nelson Academy, Ohio
Mr. James B. Francis, Civil Engineer, Lowell, Mass.
Prof. John F. Frazer, Univ. of Penn., Philadelphia.
Prof. John R. French, Prin. Mexico Academy, N. Y.
*Prof. Edward T. Fristoe, Columbian Coll., Washington, D. C.
Prof. H. R. Geiger, Wittenburg Coll., Springfield, Ohio.
Prof. W. M. Gillespie, Union Coll., Schenectady, N. Y.
*Dr. B. A. Gould, Jr., Director Dudley Obs., Albany, N. Y.
Prof. Dascom Greene, Rensselaer Institute, Troy, N. Y.
*Prof. Charles W. Hackley, Columbia Coll., N. Y.
*Mr. D. B. Hagar, Prin. Eliot High School, Jamaica Plain, Mass.
*Prof. R. D. Hamilton, Hobart Free Coll., Geneva, N. Y.
Prof. B. S. Head, Missouri Univ., Columbia
*Mr. J. B. Henck, Civil Engineer, Boston.
*Prof. W. D. Henkle, Richmond, Indiana.
*Prof. Joseph Henry, Sec'y Smithsonian Inst., Washington, D. C.
Mr. Edward C. Herrick, Librarian Yale Coll., New Haven, Conn.
Prof. Robert A. Higgins, Semple-Broaddus Coll., Centre Hill, Miss.
*Mr. J. E. Hilgard, U. S. Coast Survey, Washington, D. C.
*Rev. Thomas Hill, Waltham, Mass.
Prof. J. M. Hensley, King's Coll., Windsor, Nova Scotia.
Prof. W. Rogers Hopkins, Naval Academy, Annapolis, Md.
Prof. E. N. Horsford, Lawrence Scientific School, Cambridge, Mass.
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Mr. F. W. Hurtt, Cincinnati, Ohio.
Prof. Alonzo Jackman, Norwich Univ., Norwich, Vt.
Prof. I. W. Jackson, Union Coll., Schenectady, N. Y.
*Mr. J. F. Kellogg, Providence Conference Seminary, East Greenwich, R. I.

*Prof. E. OTIS KENDALL, Univ. of Penn., Philadelphia.
*Prof. W. C. KERR, Davidson Coll., N. C.
Prof. DANIEL KIRKWOOD, Indiana Univ., Bloomington.
Prof. BENJAMIN L. LANG, Kenyon Coll., Gambier, Ohio.
Prof. HARVEY B. LANE, Wesleyan University, Middletown, Conn.
Prof. N. R. LEONARD, Yellow Springs Coll., Iowa.
*Prof. EDMUND LONGLEY, Emory & Henry Coll., Virginia.
Mr. E. J. LOOMIS, Naut. Alm. Office, Cambridge, Mass.
Prof. ELIAS LOOMIS, New York University, N. Y.
Prof. JOSEPH LOVERING, Harvard Coll., Cambridge, Mass.
Prof. M. LYFORD, Waterville Coll., Maine.
Prof. D. H. MAHAN, Military Academy, West Point.
*Mr. JAMES MAIN, U. S. Coast Survey, Washington, D. C.
Prof. J. P. MARSHALL, Tufts Coll., Medford, Mass.
Rev. Dr. McCAWLEY, President King's Coll., Windsor, Nova Scotia.
Mr. ABIJAH McLEAN, New Lisbon, Columbia Co., Ohio.
*Mr. L. W. MEECH, U. S. Coast Survey, Preston, Conn.
Prof. J. L. MEIGS, LaGrange Synodical Coll., Tenn.
President ROBERT MILLIGAN, Kentucky University, Harrodsburg.
Prof. O. M. MITCHELL, Director Obs., Cincinnati, Ohio.
Prof. R. M. MOORE, M'Kendree Coll., Lebanon, Ill.
Mr. E. G. MORROW, Naut. Alm. Office, Cambridge, Mass.
Prof. A. L. NELSON, Washington Coll., Lexington, Va.
President C. K. NELSON, St. John's Coll., Annapolis, Md.
*Mr. SIMON NEWCOMB, Naut. Alm. Office, Cambridge, Mass.
*Prof. W. W. NEWMAN, Buffalo, New York.
*Prof. H. A. NEWTON, Yale Coll., New Haven, Conn.
Mr. W. L. NICHOLSON, U. S. Coast Survey, Washington, D. C.
*Mr. JAMES E. OLIVER, Naut. Alm. Office, Lynn, Mass.
Prof. DENISON OLMSTED, Yale Coll., New Haven, Conn.
Prof. L. M. OSBORN, Madison University, Hamilton, N. Y.
Prof. W. H. PARKER, Middlebury Coll., Vermont.
Mr. JOHN PATERSON, Albany, N. Y.
Prof. J. W. PATTERSON, Dartmouth Coll., Hanover, N. H.
*Prof. BENJAMIN PEIRCE, Harvard Coll., Cambridge, Mass.
Mr. JAMES MILLS PEIRCE, Tutor in Harvard Coll. Cambridge, Mass.
*Prof. GEORGE R. PERKINS, Utica, N. Y.
Dr. C. H. F. PETERS, Director of Obs., Hamilton Coll., Clinton, N. Y.
Prof. McKENDREE PETTY, Univ. of Vermont, Burlington.
Prof. WM. F. PHELPS, Prin. Normal School, Trenton, N. J.
*Prof. CHARLES PHILLIPS, Univ. of North Carolina, Chapel Hill.
Prof. JAMES PHILLIPS, Univ. of North Carolina, Chapel Hill.
Mr. PIKE POWERS, Staunton, Va.
Prof. J. M. PUGH, Madison Coll., Sharon Miss.
Hon. S. S. RANDALL, Supt. Public Instruction, New York City.
Prof. J. J. REYNOLDS, Washington Univ., St. Louis, Mo.
President WERDEN REYNOLDS, Worcester Female College, Worcester, Mass.
Mr. H. W. RICHARDSON, Tutor in Waterville Coll., Maine.
*Prof. J. M. RICHARDSON, Bowdon, Georgia.
Prof. H. N. ROBINSON, Elbridge, Onondaga Co., N. Y.
Prof. FAIRMAN ROGERS, Univ. of Penn., Philadelphia.
Prof. WM. B. ROGERS, Boston.
*Prof. O. ROOT, Hamilton Coll., Clinton, N. Y.
Prof. J. J. RUCKER, Georgetown Coll., Kentucky.
Prof. J. W. RUST, Bethel Female Coll., Hopkinsville, Ky.
Prof. ALFRED RYORS, Centre Coll., Danville, Kentucky.
*Mr. TRUMAN HENRY SAFFORD, Naut. Alm. Office, Cambridge, Mass.
Mr. A. SCHUYLER, Republic, Seneca Co., Ohio.
Prof. BENEDICT SESTINI, Georgetown Coll., D. C.
*Mr. THOMAS SHERWIN, Prin. English High School, Boston.
Dr. A. W. SMITH, Middletown, Conn.

Col. Francis H. Smith, Supt. Virginia Military Institute. Lexington.
Prof. F. H. Smith, Univ. of Virginia, Charlottsville.
Prof. E. S. Snell, Amherst Coll., Amherst, Mass.
*Mr. Charles H. Sprague, Naut. Alm. Office, Malden, Mass.
*Prof. M. C. Stevens, Haverford Coll., West Haverford, Pennsylvania.
President W. M. Stewart, Stewart Coll., Clarksville, Tenn.
Prof. J. F. Stoddard, Montrose, Susquehanna Co., Penn.
Prof. T. Strong, Rutgers Coll., New Brunswick, N. J.
Prof. John Tatlock, Williams Coll., Williamstown, Mass.
Prof. John Towler, Hobart Free Coll., Geneva, N. Y.
*Prof. W. P. Trowbridge, U. S. Coast Survey, Washington, D. C.
*Prof. J. M. Van Vleck, Middletown, Conn.
*Mr. G. B. Vose, U. S. Coast Survey, Washington, D. C.
Prof. W. T. Walters, Wake Forest Coll., Forestville, N. C.
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*Mr. William Watson, Instructor, Lawrence Scientific School.
*President N. B. Webster, Va. Collegiate Institute, Portsmouth, Virginia.
Prof. Henry H. White, Kentucky Univ., Harrodsburg.
*Prof. Geo. C. Whitlock, Victoria Coll., Coburg, Canada West.
Lieut. Wm. H. Willcox, U. S. N., Naval Academy, Annapolis, Md.
Prof. J. C. Wills, Randolph-Macon Coll., Virginia.
Prof. William Wines, Stewart Coll., Clarksville, Tenn.
*Prof. Joseph Winlock, Sup't Am. Naut. Alm., Cambridge, Mass.
Prof. Devolson Wood, Univ. of Michigan, Ann Arbor.
Prof. J. S. Woodman, Dartmouth Coll., Hanover, N. H.
*Mr. Chauncey Wright, Naut. Alm. Office, Cambridge, Mass.
Prof. Charles A. Young, Western Reserve Coll., Hudson, Ohio.
*Prof. W. H. Young, Ohio Univ., Athens.
Prof. Ira Young, Dartmouth Coll., Hanover, N. H.

---

## ADDITIONAL NAMES.

Prof. Ira W. Allen, Antioch Coll., Yellow Springs, Ohio.
President John Barker, Allegheny Coll., Meadville, Penn.
*Mr. W. P. G. Bartlett, Naut. Alm. Office, Cambridge, Mass.
Mr. Samuel P. Bates, Fountain Side, Meadville, Penn.
Mr. George N. Bigelow, Prin. Normal School, Framingham, Mass.
Mr. James G. Board, Hendrick's Store, Bradford Co., Va.
Prof. W. R. Boyers, Principal Buffalo Academy and Seminary, Buffalo, Putnam Co., Va.
Mr. E. E. Bradbury, Prin. High School, Ware, Mass.
Prof. John B. Cary, Principal Hampton Academy, Hampton, Va.
Prof. C. W. H. Cathcort, Dayton, Ohio.
Mr. Pliny Earle Chase, Philadelphia, Penn.
Mr. Marshall Conant, Prin. Normal School, Bridgewater, Mass.
Prof. Dorsey Cox, Burlington Coll., N. J.
Prof. M. A. Cummings, New Hampton Literary and Theological Institution, Fairfax, Vt.
Prof. Sylvester Dixon, N. H. Conference Seminary, Sanbornton Bridge, N. H.
Mr. William Edson, Civil Engineer, Boston, Mass.
Prof. Richard Ellis, Hudson River Institute, Claverack, N. Y.
Mr. Joseph Ficklin, Jr., Prin. High School, Trenton, Mo.
Mr. J. Foster Flagg, Civil Engineer, Washington, D. C.
Prof. Lewis R. Gibbes, Coll. of Charleston, S. C.
Prof. B. Franklin Greene, Director Rensselaer Polytechnic Institute, Troy, N. Y.
Mr. S. P. Hale, Prin. High School, Mt. Harmony, Tenn.
*Prof. Benjamin Hallowell, Alexandria, Va.
Mr. Sylvanus Hayward, Pembroke, N. H.

Prof. B. S. HEDRICK, New York City.
*Dr. JOEL E. HENDRICKS, Newville, Indiana.
Prof. FREDERICK HUMPHREY, State Univ., Iowa City.
Mr. WILLIAM H. JOHNSON, Supt. Schools, Buckingham, Bucks Co., Penn.
Prof. C. H. JUDSON, Furman Univ., Greenville, S. C.
Mr. ERNST KRAUSS, North Granville, N. Y.
Prof. C. W. LANE, Oglethorpe Univ., Talmage, Ga.
Mr. J. HOMER LANE, Washington, D. C.
Prof. JAMES H. MAGOFFIN, Stewart Coll., Clarksville, Tenn.
Prof. ALFRED M. MAYER, Baltimore Coll., Baltimore, Md.
Mr. JAMES McCLUNG, Tipton, Cedar Co., Iowa.
Prof. R. W. McFARLAND, Miami Univ., Oxford, Ohio.
Prof. HENRY MILES, Univ. of Bishop's Coll., Lennoxville, Canada East.
Col. E. W. MORGAN, Supt. Military Institute, near Frankfort, Ky.
Mr. F. E. PAGE, F. B. School, Providence, R. I.
Prof. J. C. PORTER, Pittsburg, Pa.
Prof. E. T. QUIMBY, Prin. Appleton Academy, New Ipswich, N. H.
Mr. SMITH RAGSDALE, Clarksville, Texas.
Hon. ANDREW J. RICKOFF, Supt. Schools, Cincinnati, Ohio.
Prof. J. L. RIDDELL, Univ. of Louisiana, N. O.
Mr. IRA SAYLES, School Commissioner, 1st Dist., Alleghany Co., Rushford, Penn.
Lieut. J. M. SCHOFIELD, Military Academy, West Point.
THOMAS W. SILLOWAY, Esq., Architect of the new Capitol, at Montpelier, Vt.
Prof. WILLIAM SMYTH, Bowdoin Coll., Brunswick, Maine.
Mr. ZELOTES TRUESDEL, Principal Union High School, Moline, Ill.
Prof. W. J. VAUGHN, Univ. of Alabama, Tuscaloosa.
Hon. W. H. WELLS, Supt. Schools, Chicago, Ill.
Prof. D. W. C. WILLIAMS, Miss. Coll., Clinton, Miss.
Mr. SOLOMON WRIGHT, Lahaska, Bucks' Co., Penn.
Prof. JEFFRIES WYMAN, Harvard Coll., Cambridge, Mass.

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We had intended to give such extracts from these replies as would indicate their nature; but are reluctantly compelled, from their number and length, to substitute the following summary:—

To the question, "Do you think there is a present need of a mathematical journal of any kind?" the replies are unanimous in the affirmative. To the question, "Do your views coincide with those here expressed as to its character?" the replies are unanimous in the affirmative. To the question, "Are you willing to assist in establishing and sustaining a journal, by contributing to its pages?" the replies, with very few exceptions, promise such aid as may be consistent with other duties; while those whose names are marked with a star have pledged constant and active coöperation.

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The following Report and Resolution were unanimously adopted in the Section of Mathematics and Physics, at the meeting of the American Association for the Advancement of Science, held at Baltimore.

"The Committee to whom was referred Mr. RUNKLE's project for the establishment of a Mathematical Journal, have examined the subject, and submit their Report as comprised in the following Resolution, which they recommend to the adoption of the Section.

"*Resolved*, That we deem the establishment of a Mathematical Journal upon the plan proposed by Mr. J. D. RUNKLE, if well executed, to be highly important to the mathematical progress of the country, and the advancement of science; and that we have full confidence in Mr. RUNKLE to do justice to the task which he has undertaken.

A. CASWELL,
BENJAMIN PEIRCE,
GEO. W. COAKLAY."

"*April* 30, 1858."

---

At the late meeting of the New Hampshire State Teachers' Association, held at Concord, the following Resolution, introduced by Prof. J. W. PATTERSON, of Dartmouth College, was unanimously adopted.

"*Resolved*, That we feel a deep interest in the establishment of a Mathematical Journal of such a character as that proposed by Mr. J. D. RUNKLE, of Cambridge, and that we will do all in our power to favor and sustain its publication, believing as we do, that it will tend greatly to promote the progress of pure and mixed mathematics, and to the advancement of the science."

---

At the recent meeting of the Iowa State Teachers' Association, the following Resolution was adopted.

"*Resolved*, That this Association cordially indorses the project of establishing a Mathematical Journal on the plan proposed by Mr. J. D. RUNKLE, of Cambridge, and commends it to the favorable consideration and support of the Teachers of Iowa.

G. W. HOUGH,
F. HUMPHREY,
J. McCLUNG,
} *Committee*."

# RUNKLE'S PRIZES

TO THE STUDENTS IN ANY INSTITUTION OF LEARNING IN THE UNITED STATES OR BRITISH PROVINCES.

## I. For Solutions.

*Judges.*

Prof. Joseph Winlock, Mr. Chauncey Wright,
Mr. Truman Henry Safford.

In the first and each succeeding number of the Journal during the year, we shall publish five problems, entitled *Prize Problems for Students.* The student who shall send us the best solutions of the greatest number of the prize problems, in any number of the Journal, in time for the third number following the one in which they are proposed, shall be entitled to a first prize of *ten dollars;* the second in order of merit shall be entitled to a second prize of *a bound copy of the first volume of the Journal.* Each solution will be estimated by the judges independently, and to the students who have the first and second highest aggregates of marks shall be awarded the prizes. The best solution of each problem will be published, entitled *Prize Solution,* with such other solutions as the judges shall consider of sufficient merit. All the steps in each solution must be fully given, and the whole communicated in a plain and legible handwriting, to secure attention. The award of the judges, announcing the names of the successful competitors, the prize solutions with the names of the authors, and credit for all solutions received, will be published in the fourth and succeeding numbers. No student shall be entitled to the same prize twice during the same year; but full credit will be given him in the award of the judges.

## II. For Essays.

*Judges.*

Prof. W. Ferrel, Mr. J. B. Henck,
Prof. George R. Perkins.

A first prize of *fifty dollars* for the best essay.

A second prize of *forty dollars* for the next in order of merit.

A third prize of *thirty dollars* for the third in order.

A fourth prize of *twenty dollars* for the fourth.

A fifth prize of *ten dollars* for the fifth.

These prizes will be given for essays upon any questions in pure or applied mathematics, including those questions in physics "which can be solved only by an application of mathematical logic to the fundamental principles which constitute the scientific conception of the phenomena."

The essays are not to be simply descriptive, but thoroughly digested discussions of the questions under consideration, presented with clearness and symmetry. Originality in matter or treatment, or in both, is certainly desirable; but even without these, an essay may show such a complete mastery of the subject as to be highly meritorious.

Essays must be brief, not exceeding in length eight pages of the Journal, and must be judged worthy of publication to be entitled to any prize.

They must be received at or before the date of publication of the eighth number, that the award of the judges may be announced, and the essays published in the first volume.

THE

# MATHEMATICAL MONTHLY.

Vol. I....OCTOBER, 1858....No. I.

## POSTSCRIPT TO INTRODUCTORY NOTE.

We have included the foregoing pages in our first Number to save any repetition that might otherwise be necessary for those who have not had their attention called to the subject; and also to put in more permanent form all that relates to the history of the enterprise. The whole subject of the establishment of the Mathematical Monthly, including its proposed character, its aims and ends, and the proper means to accomplish them, and its educational as well as scientific bearing has been so carefully considered by so large a number of those whose opinions are entitled to the greatest respect, that nothing more remains for us, in this connection, than to add, that we shall endeavor to carry out the enterprise in the spirit of its conception, and to exercise our best judgment in the use of the material placed at our disposal. We may not, however, omit this opportunity to express our sincere thanks to the many in all parts of the country, to the press as well as to individuals, for the active interest taken in the enterprise, and to acknowledge that to this influence our present favorable prospects are mainly due. We wish to be considered responsible only for such communications as appear anonymously in the Monthly.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

FIND $\theta$ from each of the equations,—

$$\tan\theta\tan 2\theta + \cot\theta = 2,$$

$$2\sin^2 3\theta + \sin^2 6\theta = 2,$$

$$\cos n\theta + \cos(n-2)\theta = \cos\theta.$$

### II.

The whole surface of a cone is three times the area of the base. Find its vertical angle.

### III.

The sum of the squares of the reciprocals of two radii vectores from the centre of an ellipse, at right angles to each other, is constant; the perpendicular from the centre, on the chord joining their extremities, is also constant. What part of the area of the ellipse is the circle whose radius is this perpendicular?

### IV.

Two circles whose radii are $R$ and $r$ touch each other externally. If $\theta$ is the angle included between the common tangents to the two circles, prove that

$$\sin\theta = \frac{4(R-r)\sqrt{Rr}}{(R+r)^2}.$$

### V.

Four circles may be described, each of which shall touch the three sides of a plane triangle or those sides produced. If six straight lines be drawn joining the centres of these circles, two and two, prove that the middle points of these six lines are in the circumference of a circle circumscribing the given triangle.

The solutions of these problems must be received by the 10th of December, in order that the awards of the judges may be announced in the number for January, 1859. See page xi. of the Introduction.

Competitors will send their names, with the names and localities of the institutions with which they are connected, on a separate slip, to be retained by the editor until the awards are made. These slips must also be signed by their Instructors, as evidence that the parties are fairly entitled to compete for the prizes.

Essays must be received by May 1st, 1859, and competitors for these prizes will comply with the directions just specified for problems. Also see page xii. of the Introduction.

We respectfully invite the attention of teachers of mathematics to the subject, and ask them to coöperate with us in giving these prizes a fair trial. If the good resulting shall be found at all in proportion to the outlay involved, some means will be found to continue them beyond our first year.

## NOTE ON DECIMAL FRACTIONS.

BY REV. THOMAS HILL.

We are convinced, both from theoretical considerations, and from experiment in the school-room, that a child should learn the use of decimal fractions as soon as he does that of whole numbers. In order that he may do this, the analogy between decimal and vulgar fractions must, at first, be kept sedulously out of sight; and the decimal fraction be treated as a natural and necessary extension of the decimal law of notation, to the right hand of the unit's place. Unless the child is thus early familiarized with the use of decimals, they never become, as they should to all arithmeticians, the most natural and easy mode of treating quantities, not simple multiples of the unit.

## RULE FOR FINDING THE GREATEST COMMON DIVISOR.

BY PLINY EARLE CHASE.

When the greatest common measure of more than two quantities is required, a rule like the following is usually given: —

"First, find the greatest common measure of any two of the numbers, then find the greatest common measure of the number found and another of the given numbers, and thus proceed till all the given numbers are brought in."

There is usually one or more of the given numbers that can be readily resolved by inspection into submultiples, and the divisor sought can thus be found very readily. But if the prime factors of the numbers are all large, the following rule is more expeditious than any other one that I have ever seen.

1. Divide all the given numbers by the least of them, and bring down the remainders.

2. Divide the first divisor and all of the first remainders by the least of them, and bring down the remainders.

3. Proceed in this manner until a remainder is found that will measure all the other remainders, and the divisor last used, — and this will be the greatest common divisor.

Every abbreviation that can be used at any step of the process, such as rejecting factors that are evidently not common, &c., should of course be employed.

### EXAMPLES.

1. Find the greatest common measure of 940, 747, 529, and 551. By inspection we resolve 940 into the factors $2^2.\,5.\,47$; $2^2.\,5.$ being evidently not common, are rejected, and we ascertain by a single trial that 47 is not a common factor. The answer is therefore 1.

2. Required the greatest common divisor of 1633, 3763, 4757, and 4189. Proceed, according to our rule, as follows: —

| | | |
|---|---|---|
| 1633)3763 | 4757 | 4189 |
| 3266 | 3266 | 3266 |
| 497 | 1491 | 923 |
| 497)1633 | 1491 | 923 |
| 1491 | 1491 | 497 |
| $142 = 2 \times 71$ | | $426 = 6 \times 71$ |
| 71)497 | | |
| 497 | | *Ans.* 71. |

## NOTE ON EQUATION OF PAYMENTS.

BY G. P. BOND.

The time at which two or more accounts, bearing interest from different dates, may be settled by a single payment of a sum equal to the total amount of all the debts, is found, according to the rule commonly used, in the following way.

*Multiply each debt by the time that must intervene before it becomes due, and divide the sum of the products by the sum of the debts. The quotient will be the interval of time required.*

If we wish to find the distance of the centre of gravity of a number of weights suspended on a straight rod, measured from a given point in the rod, we *multiply each weight by its distance from this point, and divide the sum of the products by the sum of the weights. The quotient will be the distance required.*

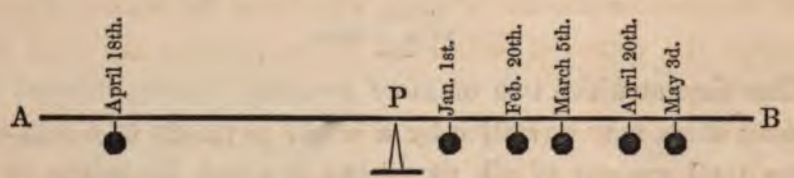
The analogy between the two processes suggests an easy mechanical method of computing the equation of payments, which we will illustrate by an example.

A merchant owes the following sums, and requires to know the

time at which, by a single payment equal to the sum of the several debts, all the accounts will be settled with interest.

| | Debts. | Bearing Interest from |
|---|---|---|
| | $500 | Jan. 1, |
| | 260 | Feb. 20, |
| | 110 | March 5, |
| | 50 | April 20, |
| | 5000 | May 3. |
| Total, | 5920 | |

In the annexed figure, A B is a bar of wood or metal balanced at P, and graduated with equal divisions to the months and days of one or more years, on each side of P.




At the graduation corresponding to Jan. 1st is hung a weight representing $500, at the date Feb. 20th another of $260, and so on, representing each sum by a proportional weight suspended from the bar at the proper dates. On the opposite side of P is hung a weight equal to the sum of all the other weights. The date (in this example, April 18th), at which it must be placed in order to restore the balance of the bar, is the time when the payment of the total sum of $5920 will discharge all the debts with interest.

The chief difficulty with the apparatus is to apportion the weights, but no great nicety will be needed, especially as fractional parts of a day and the difference between discount and interest paid in advance, are commonly disregarded in such settlements.

## ON THE STUDY OF GEOMETRY.

BY SAMUEL P. BATES.

WE propose in this paper to throw out a hint concerning the study of Geometry, which has escaped the notice of authors on this science, though it may have received the attention of teachers. Our purpose is to present matter which is practical, rather than that which is profound.

The object of a liberal course of study in the mathematics is to develop the faculties of abstraction, reason, and imagination. Few, of the many who pursue these studies, ever make any practical use of them, beyond the discipline and strength thereby acquired. It should therefore be the aim of those who direct the studies of pupils, to arrange the course of instruction so as to secure the most complete development.

Most of those who study geometry as it is taught in our seminaries and colleges, simply aim at acquiring a just conception of the several steps in the demonstrations, as they are set down in the text-book. This course, if thoroughly pursued, will vastly strengthen the reasoning powers. But from this alone does not result that high order of discipline, which ought to be gained by the study of this noble science.

The pupil should, in addition, be required to construct the series of syllogisms necessary to establish the demonstration, from that which enunciates the theorem, back to the axioms or demonstrated propositions on which the proof rests. This would serve to quicken the powers of analysis and generalization, in which consists richness of thought. The bearing of each step in the demonstration would thus be clearly discovered, and the beauty of the proof would be apparent.

But learning the demonstration of a proposition as it is given in the book, or even constructing the syllogisms which are implied in the proof, will not improve the imagination. Hence the pupil should not only be required to learn the demonstrations given, but to construct them for himself. A text-book properly prepared should furnish the demonstrations of only a portion of the propositions; the remainder should be enunciated, and the pupil required to exercise his ingenuity in discovering the proof of them.

I will illustrate my idea with an example. Suppose the text-book to furnish the demonstration of the celebrated forty-seventh proposition of Euclid, and following it the enunciation of this

*Theorem.* The triangles, formed by joining the exterior angles of the squares erected on the three sides of a right-angled triangle, are equivalent.

It is asserted that the three triangles thus formed are equivalent; it is required to prove it. The imagination is now set to work. The inquiry is at once suggested, May not the three triangles thus formed be each equivalent to the original triangle? If they are, and we can succeed in proving it, our object is attained. Some such inquiry as this must first be started. The imagination reaches out for an hypothesis; and this process must be continued until we have a theory which we can prove. We may discover the correct theory, and still not be able to construct the demonstration; but we can never construct the demonstration till we have the theory. Franklin, in his discoveries in electricity, was obliged to start with the inquiry, Is not the electrical spark identical with the flashes of lightning? This was his hypothesis, and, after repeated experiments, he succeeded in proving it true.

We thus perceive that the same faculties are required for the discovery of a geometrical demonstration as are employed in making discoveries in any of the sciences. If pupils were required to con-

struct demonstrations for themselves, while pursuing the study of geometry, their powers of original investigation would thereby be greatly improved. We sincerely hope that teachers may not only talk about this subject in their lecture rooms, but may make a practical application of it in their classes.

## DEMONSTRATION OF A THEOREM.

BY PIKE POWERS.

*"Two regular polygons, of the same number of sides, can always be formed, the one circumscribed about a circle, the other inscribed in it, which shall differ from each other by less than any assignable quantity."*

Let $A =$ area of any inscribed regular polygon,
$B =$ area of a similar circumscribed polygon,
$A' =$ area of inscribed polygon of twice as many sides,
$B' =$ area of circumscribed polygon of twice as many sides.

Then $A' = \sqrt{A.B}$, $B' = \frac{2\,A.B}{A+A'}$, (see Davies' Legendre).

Hence $$B' - A' = \frac{2\,A.B}{A+A'} - A' = \frac{2\,A\,B - A\,A' - A'^2}{A+A'}$$
$$= \frac{A\,B - A\,A'}{A+A'} = \frac{A}{A+A'}(B - A').$$

Now $A' > A \therefore B - A' < B - A, \frac{A}{A+A'} < \frac{A}{2\,A} < \frac{1}{2}$,

and $$\frac{A}{A+A'}(B - A') < \tfrac{1}{2}(B - A).$$

Hence, $$B' - A' < \tfrac{1}{2}(B - A).$$

In like manner $B'' - A'' < \frac{1}{2}(B' - A') < \frac{1}{2^2}(B - A)$,

and so on to $$B^n - A^n < \tfrac{1}{2^n}(B - A),$$

where $B^n - A^n$, the difference between the inscribed and circumscribed polygons, is infinitely small, when $n$ is infinitely great.

## DEMONSTRATION OF THE PYTHAGOREAN PROPOSITION.

BY JAMES EDWARD OLIVER.

*The square described on the hypothenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides.*

Drop a perpendicular from the right angle to the hypothenuse, and prove in the usual way that the two partial triangles thus formed are similar to the original one. On the hypothenuses of these three triangles, that is, on the three sides of the original triangle, as homologous sides, describe three figures similar to each other (squares for instance). Their areas are evidently equimultiples of the respective triangles; but two of the triangles are seen to make up the third; hence, the two corresponding squares will together make up the square on the hypothenuse. *Q. E. D.*

Perhaps the property, that similar areas are to each other as the squares of their homologous sides, being much simpler than the Pythagorean Proposition, and not depending upon it, should naturally precede it.

## NOTE ON NAPIER'S RULES.

BY TRUMAN HENRY SAFFORD.

IN the form in which they are usually given, the rules are —

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

Rule I., as will be shown, is superfluous. Represent the parts in the following order, as it were around a circle, and if $\mu$ be the middle part, $\alpha$ and $\alpha'$ the adjacent parts, $\omega$ and $\omega'$ the opposite parts, we shall have,

by Rule II., the equation

$$\sin\mu = \cos\omega' \cos\omega . \qquad (1)$$

But if $\omega'$ be the middle part, $\omega$ and $\alpha'$ will be adjacent parts, $\mu$ and $\alpha$ opposite parts, and by Rule II.,

$$\cos\mu \cos\alpha = \sin\omega'. \qquad (2)$$

If we add together the squares of (1) and (2), we shall have

$$\sin^2\mu + \cos^2\mu \cos^2\alpha = \cos^2\omega \cos^2\omega' + \sin^2\omega' ;$$

or

$$1 - \cos^2\mu \sin^2\alpha = 1 - \cos^2\omega \sin^2\omega' ;$$

or

$$\cos^2\mu \sin^2\alpha = \cos^2\omega' \sin^2\omega ;$$

or

$$\cos\mu \sin\alpha = \pm \cos\omega' \sin\omega. \qquad (3)$$

Equations (1), (2), and (3) together are sufficient for the determination of any two parts, two others being known; provided the sign of the second term of (3) is known. In the case of ordinary spherical triangles the sign is positive.

Dividing (3) without its sign by (2), we shall have

$$\tan\alpha = \cot\omega' \sin\omega ;$$

or

$$\sin\omega = \tan\alpha \tan\omega'. \qquad (4)$$

If $\omega$ be the middle part, $\alpha$ and $\omega'$ will be adjacent parts, so that Rule I. is the same as Equation (4), a deduction from Rule II. alone.

In Napier's rules the parts are, the two legs, the complements of the hypothenuse and two of the angles; the right angle not being counted.

## PROBLEM, BY PROF. PEIRCE.

What curve is represented by the equation

$$\sin\varepsilon = a\,{}^*\mathrm{Cos}\,(\log r + b),$$

in which $r$ denotes the radius vector, and $\varepsilon$ the angle between $r$ and the corresponding tangent.

* Whenever in trigonometric notation, we shall use initial capital letters, potential functions will be indicated.

## ON THE RELATION BETWEEN THE STATES OF MINIMUM AND EQUILIBRIUM.

BY JOHN PATTERSON.

1. The phenomenon of minimum is one of stable equilibrium, that of maximum being in general unstable. In the same plane, the shortest route from the point $A$ (fig. 1) to the point $B$, by the way of the line $L L'$, is well known to be through the point $P$ on the given line, where the angles $A P M$ and $B P M$ of the two branches of the path with the normal $P M$, as also the angles $A P L$ and $B P L'$ of the same branches with the line $L L'$, are respectively equal.

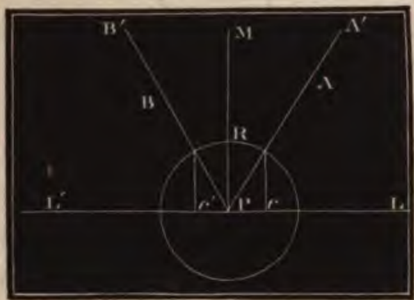


Fig. 1.

This is the familiar example of the reflection of light from a plane surface. Pass now to the dynamical form of the phenomenon. The line $L L'$ is rigid, and fixed in direction; the point $P$ is acted upon by two equal forces, in the respective directions $A'P$ and $B'P$; and as it is immaterial at what particular point in either of these rigid lines $AP$, $B P$ the respective forces originate or are applied, so long as they are equal, and act uniformly in the directions stated, their effects upon the point $P$ will be equal, and the point cannot slide upon the line, because the components of the forces in its direction are equal in magnitude and opposite in direction. With any radius $P R$ describe a circle on $P$; and the deductions from the example are,

1°. At the sought point $P$, the normal to the given line $L L'$ must make equal angles with the directions of the given origins of the equal forces; and

2°. The trigonometrical cosines $Pc$ and $Pc'$, or components of the unit measure of the respective forces on the line $LL'$, must be equal and opposed.

Then if the point $P$ be given instead of the line $LL'$, the direction of this equilibrating line will be found from the normal $PM$ obtained by dividing equally the angle $APB$.

2. Instead of two forces encountering a fixed line, the point $P$ may be held in equilibrium by any number $n$ of equal forces acting equally upon it, provided they form, two and two, equal angles about that point; for then the algebraical sum of the cosines, estimated in any direction through the point $P$, will be zero, as may be seen in the figure drawn for $n = 5$. Therefore when the points of application of five equal forces are given, the construction indicated for the point of equilibrium consists in describing on each chord $AB$, $BC$, $CD$, $DE$, $EA$ (fig. 2), an arc of circle to contain an angle of 72°; they will all intersect in the point $O$ of equilibrium, or, indeed, the intersection of any two of the circles is sufficient.

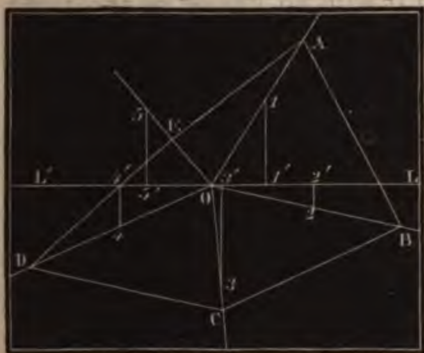


Fig. 2.

To show that $O$, the point of equilibrium, is also the point of aggregate minimum distance from the vertices of the irregular pentagon $ABCDE$ (fig. 3), draw perpendiculars $Qa$, $Qb$, $Qc$, $Qd$, $Qe$ from any other point $Q$ upon $AO$, $BO$, $CO$, $DO$, $EO$: then it is easily proved that the sum of the prolongations $Oc + Od + Oe$ is equal to the sum of the curtail-


Fig. 3.

ments $Oa + Ob$ of the lines passing from the vertices through the point $O$; whence

$$Aa + Bb + Cc + Dd + Ee = AO + BO + CO + DO + EO.$$

Now draw $AQ$, $BQ$, $CQ$, $DQ$, $EQ$; and each of these lines is the hypothenuse of a right angled triangle $AaQ$, $BbQ$, $CcQ$, $DdQ$, $EeQ$, and therefore greater than its corresponding base $Aa$, $Bb$, $Cc$, $Dd$, $Ee$. Consequently

$$AQ + BQ + CQ + DQ + EQ > AO + BO + CO + DO + EO.$$

3. The state of minimum consists, therefore, of the equilibrium of equal translatory forces upon a point: if the forces come to vary unequally, or to be arbitrarily changed in magnitude, the position of the point of equilibrium will change, and, although still remaining the equilibrating centre of the system it will no longer be the point of minimum aggregate distance simply, but the minimum of distances under the changed operation of the forces.

4. When $n = 3$, the problem requires the determination of the point of minimum aggregate distances from the three vertices $A$, $B$, $C$ of a triangle with given sides $a$, $b$, $c$. This problem was solved by VIVIANI, and published by him in his Latin treatise on *Geometrical Divinations.* The question was communicated to VIVIANI by his friend and fellow-student, TORRICELLI, to whom it appears to have been proposed by FERMAT as a sort of challenge. VIVIANI confesses in a note that the question had cost him much effort. He proved geometrically that the three angular points of the triangle must subtend equal angles at the minimum point. It is also asserted that TORRICELLI himself effected the solution by three different methods, and each differing from that of VIVIANI. Subsequently the question was discussed by many different hands, and under various aspects; and a complete general solution is given in SIMPSON'S *Treatise on Fluxions,* applicable to any number of points, and to different relations of the forces applied at them. (End in No. II.).

## NOTE ON DERIVATIVES.

BY REV. THOMAS HILL.

There are certain propositions in algebra which may be more readily demonstrated by the aid of derivatives than in any other way. As brevity and clearness of reasoning are prime requisites for a successful mathematician, I should advocate the introduction of the doctrine of derivatives at a very early stage in algebraical studies. The following form will be found, I think, sufficiently simple for young students, and sufficiently comprehensive to be used to great advantage in abbreviating modes of demonstration.

1. As an unknown quantity, $x$, may be conceived of as of any value, so it may also be conceived as increasing or diminishing so as to pass successively through various values. A quantity thus conceived of as changing its values, is called a variable.

2. Every algebraic expression containing $x$ is called a function of $x$.

3. If a variable, $x$, be multiplied by a constant, $a$, the product, $a\,x$, will manifestly change $a$ times as fast as $x$ changes. The ratio between the rate of change in a function and that in the variable, is called the derivative of the function. Thus $a$ is the derivative of the product $a\,x$. But if $a$ were the variable, the derivative relatively to $a$ would be $x$; that is $a\,x$ would change $x$ times as fast as $a$ changed.

4. If $x$ were a quantity dependent on $y$ (for instance if $x = 2\,y$), the derivative of $a\,x$ would be $a$ times the derivative of $x$, taken relatively to $y$; (that is, in the supposed case, $a$ times 2, or $2\,a$).

5. If the variable be taken $n$ times as a factor, it is manifest that the rate of change in each factor, $x$, would be multiplied by all the other $n - 1$ factors, each of which is $x$, so that the rate of change in each factor would be multiplied by $x^{n-1}$. But as there are $n$

factors, the whole rate of change in $x^n$ is $n x^{n-1}$ as fast as that in $x$. In other words the derivative of $x^n$ is $n x^{n-1}$, and of $a x^n$, it is $n a x^{n-1}$.

6. The same reasoning as that in § 5 would show, that in the $n$th power of any function of $x$, the rate of change in each function is multiplied by the other functions, and the total derivative will be the derivative of the function multiplied by $n$ times the function raised to the $(n-1)$st power.

7. $D_x$ signifies derivative relatively to $x$. Thus

$$D_x x = 1. \quad D_x a x = a. \quad D_x a x^n = n a x^{n-1}.$$

$$D_x (a + 2x)^3 = 3(a + 2x)^2 D_x (a + 2x) = 2.3(a + 2x)^2.$$

$$D_x (a + 7bx^2)^5 = 5(a + 7bx^2)^4 D_x (a + 7bx^2) = 70bx(a + 7bx^2)^4.$$

8. A derivative of a derivative is called the second derivative of the function, and these higher derivatives are denoted by exponents like those of powers.

$$D_x a x^n = n a x^{n-1},$$

$$D_x^2 a x^n = n(n-1) a x^{n-2},$$

$$D_x^3 a x^n = n(n-1)(n-2) a x^{n-3}.$$

Applications to the development of the Binomial, Taylor, and McLaurin's Theorems in the next number.

## PROPOSITIONS ON THE DISTRIBUTION OF POINTS ON A LINE.

BY PROF. BENJAMIN PEIRCE.

The following propositions are given without demonstration, in the hope that they will incite some one to a thorough investigation

of the subject, or that at least the various forms of demonstration, which will undoubtedly suggest themselves to different minds, may, if they are sent to the Journal, furnish the elements of further inquiry.

1. If two points are taken in all possible relative positions upon a given line, their distance apart in one half of the whole number of possible positions, is less than 0.29289 of the length of the line.

2. The number of positions in which the distance between the points is less than half the length of the line, is $\frac{3}{4}$ths of the whole number of possible positions.

3. The number of positions in which the distance between the points is less than one third of the length of the line, is $\frac{5}{9}$ths of the whole number of possible positions.

4. The number of positions in which the distance between the points is less than two thirds of the length of the line, is $\frac{8}{9}$ths of the whole number of possible positions.

5. If the line returns into itself like the circumference of the circle, the two points cannot be further apart than one half the length of the line, and for any given smaller distance, the number of positions, in which the distance between the points is less than this distance, is proportional to the distance.

6. If the line returns into itself, and if three points are assumed upon it, the number of positions, in which the two nearest points are at a less distance apart than one fourth of the length of the line, is $\frac{15}{16}$ths of the number of all possible positions.

7. In the preceding case, the least distance apart of the two nearest points is 0.09763 of the length of the line in one half of the whole number of possible cases.

8. In the preceding case, the three points are always included in an arc of the line, which is two thirds of the whole length;

in $\frac{3}{4}$ths of all the possible positions, they are included in half the length of the arc; the number of positions, in which the three points are included in an arc which is less than half the length, is proportional to the square of the length of the arc; the number of positions in which the three points are included in an arc which is $\frac{5}{8}$ths of the whole length is $\frac{63}{64}$ths of the whole number of positions.

## VIRTUAL VELOCITIES.

BY WILLIAM WATSON.

It is proposed to deduce the law of virtual velocities from the general equations of equilibrium.

Let $F_1$, $F_2$, $F_3$, &c., be a number of forces by which any material system is held in equilibrium; $f_1$, $f_2$, $f_3$, &c., their respective lines of direction.

Suppose this system to receive any displacement whatever, $\delta p$, consistent with the conditions to which the system is subject, such that the component of $\delta p$ due to the rotation of the system, is infinitesimal.

*Then the algebraic sum of the products, obtained by multiplying the intensity of each force by the correspondiug distance which its point of application advances in the direction of that force, must be equal to nothing;* that is,

$$F_1 \delta f_1 + F_2 \delta f_2 + F_3 \delta f_3 + \&c. \ldots = 0;$$

which may be written $\Sigma F \delta f = 0.$

The distance $\delta f$ is called the virtual velocity with reference to the force, $F$.

The term velocity was first employed because the displacements, which were taken infinitely small, were conceived to be made in

the *same* infinitesimal time, and therefore the spaces were proportional to the velocities.

The term virtual was used to show that the displacements were hypothetical, and not actual. The term virtual moment is usually applied to the expression $F\delta f$.

The principle may be concisely stated thus; if any system is in equilibrium, the sum of the virtual moments of all the forces vanishes.

Let the components of $\delta p$ be a translation $\delta q$, and a rotation $d\theta$ about any axis. This rotation is equivalent to three simultaneous rotations $d\theta_x$, $d\theta_y$, $d\theta_z$, about three rectangular axes $X, Y, Z$; therefore,
$$d\theta = \sqrt{(d\theta_x^2 + d\theta_y^2 + d\theta_z^2)}.$$

Resolving $\delta q$ parallel to the axes $X$, $Y$, $Z$, and calling the respective components $\delta q_x$, $\delta q_y$, $\delta q_z$, we have
$$\delta q = \sqrt{(\delta q_x^2 + \delta q_y^2 + \delta q_z^2)}$$

Now find how much a point, whose coördinates are $(x\,y\,z)$, advances along the three axes of coördinates in virtue of the elementary rotation $d\theta$.

Let fall from $(x\,y\,z)$ a perpendicular $\varrho_z$ upon the axis $Z$. Then the angle, which the arc $\varrho_z d\theta_z$ makes with $X$, is the same as that which $\varrho_z$ makes with $Y$, that is, $\cos^{-1}\frac{y}{\varrho_z}$. Therefore, projecting the arc $\varrho_z d\theta_z$ upon $X$ and $Y$, and considering the rotation positive, we have

$$-\varrho_z d\theta_z \frac{y}{\varrho_z} = -y\,d\theta_z = \text{motion along } X \text{ in virtue of rotation } d\theta_z;$$

$$\varrho_z d\theta_z \frac{x}{\varrho_z} = x\,d\theta_z = \text{" " } Y \text{ " " } d\theta_z.$$

Similarly we find

$$-\varrho_z d\theta_x \frac{z}{\varrho_z} = -z\,d\theta_x = \text{" " } Y \text{ " " } d\theta_x;$$

$$\varrho_z d\theta_x \frac{y}{\varrho_z} = y\,d\theta_x = \text{" " } Z \text{ " " } d\theta_x;$$

$$-\varrho_z d\theta_y \frac{x}{\varrho_z} = -x\,d\theta_y = \text{" " } Z \text{ " " } d\theta_y;$$

$$\varrho_z d\theta_y \frac{z}{\varrho_z} = z\,d\theta_y = \text{" " } X \text{ " " } d\theta_y.$$

Combining these motions with the translations, and calling $\delta x$, $\delta y$, $\delta z$, the whole distance the point advances in the direction of $X$, $Y$, $Z$, we have

$$\delta x = \delta q_x + z\,d\theta_y - y\,d\theta_z, \qquad (a)$$
$$\delta y = \delta q_y + x\,d\theta_z - z\,d\theta_x, \qquad (b)$$
$$\delta z = \delta q_z + y\,d\theta_x - x\,d\theta_y, \qquad (c)$$

Now take the equations of equilibrium

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$
$$\Sigma (Zy - Yz) = 0, \quad \Sigma (Xz - Zx) = 0, \quad \Sigma (Yx - Xy) = 0,$$

and multiply them respectively by $\delta q_x$, $\delta q_y$, $\delta q_z$, $d\theta_x$, $d\theta_y$, $d\theta_z$; adding the products, we get

$$\Sigma \{X(\delta q_x + z\,d\theta_y - y\,d\theta_z) + Y(\delta q_y + x\,d\theta_z - z\,d\theta_x)$$
$$+ Z(\delta q_z + y\,d\theta_x - x\,d\theta_y)\} = 0. \qquad (d)$$

Using $(a)$, $(b)$, $(c)$, we reduce $(d)$ to

$$\Sigma \{X\delta x + Y\delta y + Z\delta z\} = 0. \qquad (e)$$

But $$\delta x = \delta p \cos{}^{x}_{\delta p}, \quad \delta y = \delta p \cos{}^{y}_{\delta p}, \quad \delta z = \delta p \cos{}^{z}_{\delta p},$$
$$X = F\cos{}^{x}_{f}, \qquad Y = F\cos{}^{y}_{f}, \qquad Z = F\cos{}^{z}_{f};$$

substituting these values in $(e)$ it becomes

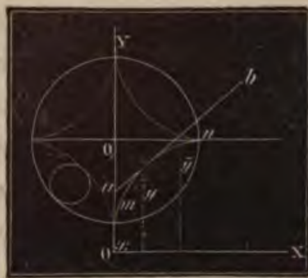
$$\Sigma \{F\delta p\,(\cos{}^{x}_{\delta p}\cos{}^{x}_{f} + \cos{}^{y}_{\delta p}\cos{}^{y}_{f} + \cos{}^{z}_{\delta p}\cos{}^{z}_{f})\} = 0;$$

or $$\Sigma \{F\delta p \cos{}^{f}_{\delta p}\} = \Sigma F\delta f = 0.$$

### PRACTICAL APPLICATION.

The principle of Virtual Velocities may be applied with advantage to the solution of the following problem, which was first proposed by WALTON.

A heavy rod $a\,b$, situated in the plane $X\,O\,Y$, rests with one end against a smooth vertical wall $O\,Y$, and upon a smooth plane curve, $m\,n$; to determine the nature of the curve that the rod may be in equilibrium in every position.



Let $2l =$ length of rod,

$(xy) =$ coördinates of the point of tangency,

$\bar{y} =$ ordinate of centre of gravity of the rod,

$$\bar{y} = y + (l - xD_x s)\,D_s y$$
$$= y - xD_x y + l\,D_s y.$$

If the rod is displaced, $\bar{y}$ becomes $\bar{y} + d\bar{y}$; but by the principle of virtual velocities $F d\bar{y} = 0$ for equilibrium, $F$ being the weight of the rod. $\therefore \quad d\bar{y} = 0 = -\,xD_x^2\,y + D_x\,l\,\frac{D_x y}{1+(D_x y)^2}$

$$= -\,xD_x^2 y + \frac{l\,D_x^2 y}{(1+(D_x y)^2)^{\frac{3}{2}}}.$$

This equation is satisfied by putting $D_x^2 y = 0$, which is the differential equation of a straight line; also by putting

$x(1 + (D_x y)^2)^{\frac{3}{2}} - l = 0$; from which $D_x y = \sqrt{(l^{\frac{2}{3}} - x^{\frac{2}{3}})}\,.\,x^{-\frac{1}{3}}$

$\therefore\ y = \int \sqrt{(l^{\frac{2}{3}} - x^{\frac{2}{3}})}\, x^{-\frac{1}{3}}\,dx + \text{const.} = -\,(l^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} + \text{const.}$

If we take for the axis of $x$ the horizontal position of the rod, then the constant disappears, and we have, after reducing, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$, which is the equation of a hypocycloid, in which $l$ is the radius of the fixed circle, and $\frac{l}{4}$ the radius of the moving circle.

## THE PRISMOIDAL FORMULA.

BY CHAUNCEY WRIGHT.

The formula $(B + 4B' + B'')\frac{l}{6}$, in which $B, B', B''$ are three equidistant parallel surfaces, sections of a solid, and $l$ the distance between $B$ and $B''$, is the expression for the solid contents between

$B$ and $B'$, not only for the prismoid from which it was first derived, but also for several solids of revolution, as the sphere, ellipsoid, &c. What is the extent of its application?

Let $X$ be the axis perpendicular to which the sections are made, and $f(x)$ the area of the section at the distance $x$ from the origin. The problem then is, What function $f$ will fulfil the conditions of the formula?

Let three sections be made through any solid at the distances $(x - h)$, $x$, $(x + h)$ from the origin. Then $h = \frac{l}{2}$; and $f(x - h)$, $f(x)$, $f(x - h)$ will be the areas of the sections, and if the formula apply the solid contents between $f(x - h)$ and $f(x + h)$ will be $[f(x - h) + 4f(x) + f(x + h)]\frac{h}{3}$. But the solid contents is also equal to the integral of the differential solid $f(x)\,dx$ between the limits $x - h$ and $x + h$.

Equating these two expressions for the solid contents, we have, if the function $f$ fulfils the conditions of the formula,

$$\int_{x-h}^{x+h} f(x)\,dx = \int f(x+h)\,dx - \int f(x-h)\,dx$$
$$= \Big[f(x-h) + 4f(x) + f(x+h)\Big]\frac{h}{3}.$$

To find what form of $f$ will satisfy this equation, develop both its members by Taylor's theorem. The first member becomes,

$$(A)\quad \left.\begin{aligned} &\int\Big[f(x) + f'(x)h + f''(x)\frac{h^2}{2} + f'''(x)\frac{h^3}{2\cdot 3} + \text{\&c.}\Big]\,dx \\ -&\int\Big[f(x) - f'(x)h + f''(x)\frac{h^2}{2} - f'''(x)\frac{h^3}{2\cdot 3} + \text{\&c.}\Big]\,dx \end{aligned}\right\}$$
$$= \int 2f'(x)h\,dx + \int 2f'''(x)\frac{h^3}{2\cdot 3}\,dx + \text{\&c.}$$
$$= 2f(x)h + f''(x)\frac{h^3}{3} + \overset{\text{IV}}{f}(x)\frac{h^5}{3\cdot 4\cdot 5} + \text{\&c.}$$

The second member becomes

$$(B)\quad \left.\begin{array}{l}\Big[f(x)-f'(x)\,h+f''(x)\frac{h^2}{1.2}-\text{\&c.}+4\,f(x)\\ +f(x)+f'(x)\,h+f''(x)\,\frac{h^2}{1.2}+\text{\&c.}\Big]\frac{h}{3}\end{array}\right\}$$

$$=\Big[6f(x)+f''(x)\,h^2+\overset{\text{IV}}{f}(x)\frac{h^4}{3.4}+\text{\&c.}\Big]\frac{h}{3}$$

$$=2f(x)\,h+f''(x)\frac{h^3}{3}+\overset{\text{IV}}{f}(x)\,\frac{h^5}{36}+\text{\&c.}$$

Comparing the last members of $(A)$ and $(B)$ we find the equation

$$2f(x)\,h+f''(x)\,\frac{h^3}{3}+\overset{\text{IV}}{f}(x)\frac{h^5}{60}+\text{\&c.}=2f(x)\,h+f''(x)\frac{h^3}{3}$$
$$+\overset{\text{IV}}{f}(x)\frac{h^5}{36}+\text{\&c.}$$

which is satisfied only when the terms beyond the second disappear; that is, by functions which have no fourth and higher derivatives. Hence $f(x)$ must be an algebraic expression of positive integral powers not exceeding the third degree.

In general

$$f(x)=a\,x^3+b\,x^2+c\,x+e=\text{area of section.}$$

The error in applying this formula, when $f(x)$ is of the fourth or fifth degree is

$$\overset{\text{IV}}{f}(x)\,h^5\left(\tfrac{1}{36}-\tfrac{1}{60}\right)=\overset{\text{IV}}{f}(x)\,\frac{h^5}{90}.$$

If $h$ is taken so small, that terms containing its fifth power may be neglected, the prismoidal formula may be applied to any solid. Simpson's rule, which includes a series of prismoidal formulas, is sufficiently accurate when $n$ is so large that $\left(\frac{l}{n}\right)^5$ may be neglected.

The prismoid comes within the limits of this formula. For, being a solid bounded by planes, and the sections perpendicular to its sides being dissimilar polygons, but with the same number of sides, it may be decomposed into prismoids with triangular sections. These prismoids are bounded by one plane and two warped surfaces; the plane

surface being one of the surfaces of the original prismoid; the other two being generated by the varying sides of the triangular section. Since these triangles are not similar, their sides must change at dissimilar rates in passing along the axis, or their base and altitude will not change proportionally.

Let $b$ be the base and $a$ the altitude of any triangle at the origin of $x$; let $\alpha$ and $\beta$ be their respective rates of change, in passing along $x$. Then $b + \beta x$ and $a + \alpha x$ will be the base and altitude at any distance $x$ from the origin, and its area will be $\frac{1}{2}(a + \alpha x)(b + \beta x)$. Hence the area of the whole polygonal section is

$$f(x) = \tfrac{1}{2}\Sigma(a + \alpha x)(b + \beta x) = \tfrac{1}{2}\Sigma\,[ab + (a\beta + b\alpha)x + \alpha\beta x^2]$$

which is an expression of the second degree, and is therefore one to which the formula applies.

The formula also applies to the solids generated by the revolution of the conic sections about their axes of symmetry, the general equation of these sections being $y^2 = mx + nx^2$. The sections of these solids taken perpendicularly to the axis of $x$ are circles, having $y$ for a radius, and $\pi m x + \pi n x^2 = \pi y^2 = f(x) =$ area, to which also the formula applies.

But if the conic section be revolved about any axis for which the general equation does not hold, then the prismoidal formula does not apply, because the expression for the area of the section $f(x)$ will involve radicals which have fourth and all higher derivatives. The formula applies in like manner to the solid of revolution of the semicubic parabola, of which the equation is $y^2 = mx^3$.

The principles involved in this discussion of the prismoidal formula, may be applied to finding a formula, for solid contents, which shall hold good for bodies, of which the sections are expressed in terms of higher degrees than the third; and also for cases in which the sections are taken at unequal intervals.

This discussion is reserved for the next number of the Monthly.

## OVALS AND THREE-CENTRE ARCHES.

BY J. B. HENCK.

An oval, $A\,B\,C\,D$ (fig. 1) is composed of two pairs of circular arcs, of which one pair $E\,F$ and $G\,H$ have their centres on the diameter $B\,D$, and the other pair $E\,G$ and $F\,H$ have their centres on the diameter $A\,C$ or on this diameter produced, the two diameters being at right angles to each other. A three-centre arch is one half, $B\,A\,D$, of an oval; so that the half oval only need be considered.

Fig. 1.

As these figures are of importance to the engineer and architect, it is proposed to give a general method of drawing the oval, when its two diameters are given, and one of the radii is assumed. A general algebraic relation between the semidiameters and the radii will also be established. The special cases derived from this general relation will serve to correct some erroneous methods of drawing ovals given in drawing books, and will also give the foundation of such methods as are correct, and the means of *computing* the radii, matters generally neglected.

### GEOMETRICAL METHOD, WHEN ONE RADIUS IS ASSUMED.

Let $B\,D$ and $A\,C$ (fig. 1) be the diameters of the required oval, and let $B\,I$ be assumed as the first radius. From $I$ as a centre describe the quadrant $B\,E\,K$. Through $A$ and $K$ draw a straight line, and produce it, till it meets the curve $A\,K$ in $E$. Through $E$ and $I$ draw the line $E\,L$, cutting $A\,C$, produced if necessary in $L$. Then $L$ is the centre of the second arc of the oval, and $E$ is the common point of tangency of the two arcs. To prove this, it will be suffi-

cient to show that $AL = EL$. Draw $KI$, and we have the triangle $AEL$ similar to the isosceles triangle $KEI$. Therefore, $AL = EL$.

If the radius $AL$ had been assumed, and with it, the quadrant $AEM$ described, the point $E$ would have been found by drawing a straight line through $M$ and $B$, and producing it till it met the arc $AM$.

It should be observed, that the first radius must always be less than $AN$, and the second radius always greater than $BN$.

### GENERAL RELATION BETWEEN THE SEMI-DIAMETERS AND THE RADII.

Let $BN = a$ (fig. 1), $AN = b$, $BI = r$ and $AL = r_1$. Then in the right triangle $INL$ we have $IL = r_1 - r$, $IN = a - r$ and $NL = r_1 - b$.

$$\therefore (r_1 - r)^2 = (a - r)^2 + (r_1 - b)^2,$$

$$\therefore r_1^2 - 2rr_1 + r^2 = a^2 - 2ar + r^2 + r_1^2 - 2br_1 + b^2,$$

$$\therefore 2ar + 2br_1 - 2rr_1 = a^2 + b^2. \qquad (1)$$

This equation expresses the relation that must always unite the four quantities $a$, $b$, $r$, and $r_1$. When any three of these are given, the fourth may be found. When $a$ and $b$ are considered fixed, the value of either radius may be assumed, and the other calculated. Thus if the first radius $r$ be assumed, (1) gives for the second radius

$$r_1 = \frac{a^2 + b^2 - 2ar}{2b - 2r} = a + \frac{(a-b)^2}{2b - 2r}; \qquad (2)$$

and if the second radius $r_1$ be assumed, (1) gives for the first radius

$$r = \frac{a^2 + b^2 - 2br_1}{2a - 2r_1} = b - \frac{(a-b)^2}{2r_1 - 2a} \qquad (3)$$

*Example.* If one diameter of an oval be 24, and the other 18, and if the first radius be assumed equal to 6, we have $a = 12$, $b = 9$, and $r = 6$, to find $r_1$. Here (2) gives for the second radius

$$r_1 = 12 + \frac{3^2}{18 - 2 \times 6} = 12 + \tfrac{9}{6} = 13.5.$$

If the second radius had been assumed $= 13.5$, the first radius would be from (3)

$$r = 9 - \frac{3^2}{2 \times 13.5 - 24} = 9 - \tfrac{9}{3} = 6.$$

The applications are reserved for the next number.

# Mathematical Monthly Notices.

*The Quarterly Journal of Pure and Applied Mathematics. Edited by* J. J. SYLVESTER, *M. A., F. R. S., late Professor of Natural Philosophy in University College, London; and* N. M. FERRERS, *M. A., Fellow of Gonville and Caius College, Cambridge: assisted by* G. G. STOKES, *M. A., F. R. S., Lucasian Professor of Mathematics in the University of Cambridge;* A. CAYLEY, *M. A., F. R. S., late Fellow of Trinity College, Cambridge; and* M. HERMITE, *Corresponding Editor in Paris.* London: JOHN W. PARKER AND SON, West Strand.

THIS journal may be considered as the continuation of the Cambridge and Dublin Mathematical Journal, the publication of which was discontinued with the ninth volume in 1854. The first number of the Quarterly Journal bears date April, 1855, and number eight, which completes the second volume, was published May, 1858. The next number will appear in October. We shall only say at present, that this Journal is an able exponent of the present state and progress of mathematical science in England.

*Nouvelles Annales de Mathématiques. Journal des Candidats aux écoles Polytechnique et Normale: Rédigé par* M. TERQUEM, *Officier de l'Université, Docteur des Sciences, Professeur aux Écoles Impériales d'Artillerie, Officier de la Légion d'honneur, et* M. GERONO, *Professeur des Mathématiques.* Paris: MALLET-BACHLIER.

This journal is issued on the first of every month, containing either 32 or 48 pages, 8vo. It was begun in 1842, and sixteen volumes are now complete. The last number received is for July, 1858. In 1855 the "Bulletin de Bibliographie, d'Histoire et de Biographie Mathématiques" was commenced, a few pages being added to each number of the journal. The "Bulletin" is paged independently, and makes an annual volume of about 200 pages, of which three are complete. This journal contains solutions and discussions of all degrees of difficulty, but its chief aim evidently is to influence the course of mathematical study, and stimulate both teachers and students to aim at a high standard of excellence; and from the character of the journal we have no doubt of the result.

*Journal für die reine und angewandte Mathematik. In zwanglosen Heften. Als Fortsetzung des von* A. C. CRELLE, *gegründeten Journals herausgegeben unter Mitwirkung der* HERREN STEINER, SCHELLBACH, KUMMER, KRONECKER, WEIRSTRASS, *von* C. W. BORCHARDT.

*Mit thätiger Beförderung hoher Königlich-Preufsischer Behörden.* Berlin: Druck und Verlag von GEORG REIMER.

Crelle's Journal was begun in 1826, and has been continued without interruption until the present time. Each volume contains from 375 to 400 quarto pages; the one for 1857 being the fifty-third of the series. The fiftieth volume, issued in 1855, contains a table of contents of the fifty volumes, and an examination of this table must convince any one of the great value of this journal. We find the papers of Abel, 24 in number, making 384 pages; 19 by Cayley, pp. 216; 44 by Crelle, pp. 1187; 26 by Lejeunne-Dirichlet, pp. 390; 37 by Eisenstein, pp. 698; 38 by Gudermann, pp. 1489; 25 by Hesse, pp. 329; 98 by Jacobi, pp. 1575, besides posthumous papers still appearing; 25 by Kummer, pp. 523; 29 by Minding, pp. 216; 26 by Möbius, pp. 361; 6 by Ottinger, pp. 1035; 12 by Olivier, pp. 121; 6 by Plana, pp. 293; 21 by Plücker, pp. 346; 37 by Raabe, pp. 413; 22 by Richelot, pp. 533; 13 by Schellbach, pp. 197; 51 by Steiner, pp. 687; 28 by Stern, pp. 400; with over 200 other contributors. In short, it would be difficult to exaggerate the value of this Journal, and we believe that no one could enter upon any investigation in the higher mathematics, before consulting Crelle, without running a great risk of having been anticipated in its pages. We sincerely hope that Professors of Mathematics and all Library Committees will remember that this journal is the very best possible foundation for a mathematical library, which cannot be complete without it.

Special notices of the labors of various contributors to the above journals are reserved for future occasions.

*Physical and Celestial Mechanics. By* BENJAMIN PEIRCE, *Perkins Professor of Astronomy and Mathematics in Harvard University, and consulting Astronomer of the American Ephemeris and Nautical Almanac. Developed in four systems of Analytic Mechanics, Celestial Mechanics, Potential Physics, and Analytic Morphology.* Boston: LITTLE, BROWN & Co. 1855.

Of this proposed course on Mechanics, the first volume, entitled A System of Analytic Mechanics, is already issued. Some twelve years ago, Prof. Peirce announced a series of three volumes, on "Curves, Functions, and Forces." Volumes I. and II., known by the name of "Curves and Functions," were then published by JAMES MUNROE & Co.; and the volume before us, although a part of a new and vastly enlarged course, may be considered as the redeeming of this former pledge.

It is unnecessary for us to say that this is Prof. Peirce's greatest work, and more than realizes our highest expectations.

Every part of the work is treated with that concise fulness for which the author is so remarkable, and which has enabled him to condense more matter into 496 quarto pages than was ever before put into the same space. In fact, we do not think of a single subject, coming strictly within the limits of the first volume of the course, left untouched. Some idea may be given of the amount the volume contains, by remarking that the table of contents covers twenty-seven closely printed double-column pages, and the alphabetical index, ten similar pages. We need not be more specific, for mathematicians will read and judge of the work for themselves. Besides, we shall have occasion to refer to its contents hereafter.

A striking feature of the volume is the style in which it is issued. We unhesitatingly pronounce it the finest specimen of mathematical printing we have ever seen, and do not believe it has ever been surpassed, if equalled, in any country. For this, the publishers deserve and receive the greatest credit. It is but justice, however, and it gives us great pleasure to add, that the work was printed by Messrs Allen & Farnham, of Cambridge, and our readers will not fail to discover the evidence of their skill in the Mathematical Monthly.

# THE

# MATHEMATICAL MONTHLY.

Vol. I....NOVEMBER, 1858....No. II.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

WHAT conditions ought the sides and angles of a parallelogram to satisfy, in order that a square may be inscribed in it?

### II.

Measure the sides $a, b, c$ of a plane triangle, and suppose that $\alpha, \beta, \gamma$ are respectively the errors of the sides. Find the influence of these errors upon the angles $A, B, C$.

### III.

Two triangles, $ABC$ and $abc$, are similar when the angles $A$ and $a$ are equal, and the sides $BC$ and $bc$, opposite the equal angles, are to each other as the perimeters of the triangles.

### IV.

Show that the product of six entire consecutive numbers cannot be the square of a commensurable number.

### V.

If, through any point, secants be drawn to a curve of the second order, and through the two points where each secant intersects the

curve, tangents be drawn meeting each other, the locus of all such points of meeting is a straight line.

The solutions of these problems must be received by the 10th of January, 1859.

Credit for the prize problems will always be given with their solutions.

## FIRST LESSONS IN NUMBER.

BY REV. THOMAS HILL.

As the senses are developed before the power of abstract thought, so concrete arithmetic should precede the arithmetic of pure numbers. The little wood-cuts, given in the most elementary books on arithmetic, are not sufficient for the purpose of conveying just ideas of number. For beginners in arithmetic, a quart of corn or beans is an almost indispensable apparatus. The sliding beads upon a wire frame, frequently used, are rendered comparatively worthless by having twelve beads upon a rod, in threes of one color. They should have ten beads upon a rod, five of each color.

One of the earliest points to which the attention of the child should be directed, is the difference between prime and composite numbers. The natural series of numbers from one to twelve, will afford food for thought and materials for amusement to a young scholar for many months. Let him take, for example, six beans, and arrange them in two groups of three each, this will show him that $2 \times 3 = 6$. Each of the three beans, in one pile, may take a companion from the other, and thus we shall have $3 \times 2 = 6$. From this example, the child may commence the induction that the value of the product is independent of the order of the factors. Dividing the six again into two threes, we may again reduce it to three twos

by simply subtracting one from each of the threes and putting these two ones together. From this the child may begin his inductions in regard to permutations and combinations. If one is now added to the whole, making seven, none of the properties of a composite number remain, but some of the truths concerning permutation and combination are unaffected.

The young child must not be expected, from his arrangement of beans upon his desk, to *deduce* the properties of numbers, but only to *see* them. It is the sense, rather than the reason, which is to be first exercised.

## SUBTRACTION BY MULTIPLICATION AND ADDITION.

BY PLINY EARLE CHASE.

THE following rule will doubtless be new and interesting to such readers of the Mathematical Monthly as are not familiar with Mann and Chase's Arithmetics, or Chase's Common School Arithmetic.

*Write nine times the subtrahend under the minuend, and add each figure of the upper number to the figures in the same place, and all the inferior places of the lower number, carrying as in ordinary addition. Proceed in this manner, stopping at the figure that falls immediately under the left hand figure of the minuend, and the result will be the difference sought.*

*Example.* Required the difference between 9047 and 23615.

$$\begin{array}{l} 23615 \\ \underline{81423} = 9 \times 9047 \\ 14568 \end{array}$$

5 and 3 are 8; 1 and 2 and 3 are 6; 6 and 4 and 2 and 3 are 15; 1 to carry and 3 and 1 and 4 and 2 and 3 are 14; 1 to carry and 2 and 8 and 1 and 4 and 2 and 3 are 21, but as the 1 falls under the left hand figure of the minuend, we stop there, and find the true remainder to be 14568.

*Demonstration.* If $M$ represents the minuend, $S$ the subtrahend,

and $x$ the number of figures in the minuend, the result obtained by our rule is

$$M+(\overline{10^{x}-1}-10^{x})\,S=M-S.$$

## NOTE ON THE EXTRACTION OF THE CUBE ROOT OF NUMBERS.

BY THE EDITOR.

In the Nouvelles Annales de Mathématiques for January, 1858, we find the following method for extracting the cube root of numbers, which ought, on account of its easy application, to be generally used. The editor remarks in the April number, that the method had previously been given in a work entitled *Calcul pratiques*, in which it is claimed as new. The reader will find the same process, entitled a new method, in the American edition of Young's Algebra, published as long ago as 1832. It may also be found in some of our arithmetics; and many teachers, undoubtedly, already know and use it.

It is well known, that, in the úsual rule, the chief difficulty consists in forming the trial divisors, and the labor increases in a rapid ratio with the number of figures in the root. The method proposed reduces this part of the process to the simple addition of numbers already used. If $r_1$, $r_2$, $r_3$, &c. denote the successive figures of the root, it is known that $r_1$ is found by simple inspection from the first, or left hand period of the given number, $N$, and that $3\,r_1^2$ is the trial divisor for finding $r_2$.

As soon as $r_2$ is found, we subtract from $N$, the cube

$$\begin{aligned}(r_1+r_2)^3&=r_1^3+3\,r_1^2r_2+3\,r_1r_2^2+r_2^3\\ (1)\qquad\qquad &=r_1^3+(3\,r_1^2+3\,r_1r_2+r_2^2)\,r_2,\end{aligned}$$

in which the quantity in the parenthesis is the complete divisor. The trial divisor for finding $r_3$, is

$$3(r_1 + r_2)^2 = 3r_1^2 + 6r_1r_2 + 3r_2^2$$

$$(2) \qquad = (3r_1^2 + 3r_1r_2 + r_2^2) + (3r_1r_2 + r_2^2) + r_2^2.$$

Now, comparing (1) and (2), we see at once, that the successive trial and complete divisors are made from each other as follows:—

(3) 1st trial divisor $= 3r_1^2$.

(4) 1st complete divisor $= 3r_1^2 + (3r_1 + r_2)r_2$.

(5) 2nd trial divisor $= 3(r_1 + r_2)^2$

$$= 3r_1^2 + (3r_1 + r_2)r_2 + (3r_1 + r_2)r_2 + r_2^2.$$

This shows, that when we have once computed $(3r_1 + r_2)r_2$ to find (4) from (3), we use the same result to find (5) from (4), only adding $r_2^2$ besides. In this way any trial divisor is made from the preceding complete divisor, by the addition of results already found.

The following example will sufficiently illustrate the method, and show the facility with which it can be used.

Extract the cube root of 162467446993496.

| | |
|---|---|
| $\dot{1}62\dot{4}67\dot{4}46\dot{9}93496$ | 54566 = root |
| 125 | $3 \times 5^2 = 75\cdot\cdot$ |
| 37467 | 616 = (3 × 5 + 4)4 |
| 32464 | 8116 × 4 |
| 5003446 | 16 |
| | $3 \times 54^2 = 8748\cdot\cdot$ |
| | 8125 = (3 × 54 + 5)5 |
| 4414625 | 882925 × 5 |
| 588821993 | 25 |
| | $3 \times 545^2 = 891075\cdot\cdot$ |
| | 98136 = (3 × 545 + 6)6 |
| 535233816 | 89205636 × 6 |
| 53588177496 | 36 |
| | $3 \times 5456^2 = 89303808\cdot\cdot$ |
| | 982116 = (3 × 5456 + 6)6 |
| 53588177496 | 8931362916 × 6 |
| | 36 |
| | $3 \times 54566^2 = 8932345068$ |

It will be observed, that in the parentheses, the numbers preceding the sign + are tens, and those after it units.

## SIMPLIFICATION OF THE EXPRESSION $\sqrt{a \pm c\sqrt{d}}$.

BY JOHN M. RICHARDSON.

THE solution of equations of the fourth degree, involving only the even powers of the unknown quantity, gives rise to the expression $\sqrt{(a \pm c\sqrt{d})}$. It can often be reduced to a simpler form, and it is proposed here to investigate it, and to deduce from the investigation a simple rule independent of any formula for the reduction of all such expressions.

The square root of a binomial cannot be extracted so long as it retains its binomial form; but if it can be resolved into a trinomial, the terms being so related that twice the product of the square roots of the extremes shall be equal to the mean, the root of the expression is equal to the sum or difference of the square roots of the extremes.

Whenever the numerical values of $a$, $c$, and $d$ are given, $a \pm c\sqrt{d}$, can easily be resolved into a trinomial; and even when the expression is purely algebraic, it can in many cases be simplified.

Now, $\qquad a \pm c\sqrt{d} = (a - d) \pm c\sqrt{d} + d,$

and if $(a - d)$ is a perfect square whose root is the half of $c$, there results

$$\sqrt{(a \pm c\sqrt{d})} = \tfrac{1}{2}c \pm \sqrt{d}.$$

Before enunciating the rule, some numerical and literal examples will be solved.

$$\sqrt{(55 \pm 14\sqrt{6})} = \sqrt{(49 \pm 14\sqrt{6} + 6)} = 7 \pm \sqrt{6}.$$

$$\sqrt{(29 - 12\sqrt{-7})} = \sqrt{[36 - 12\sqrt{-7} + (-7)]} = 6 - \sqrt{-7}.$$

$$\sqrt{(1 \pm 4\sqrt{-3})} = \sqrt{[4 \pm 4\sqrt{-3} + (-3)]} = 2 \pm \sqrt{-3}.$$

$$\sqrt{(c^2 + ab \pm 2c\sqrt{ab})} = \sqrt{(c^2 \pm 2c\sqrt{ab} + ab)} = c \pm \sqrt{ab}.$$

The rule, then, is simply this: *Subtract* d *from* a, *and if the remainder is a perfect square whose square root is one half of* c, *the root of the expression*

is $\frac{1}{2}c \pm d$. In other words, *When the expression can be simplified, the root is half the coefficient of the second radical plus or minus, the radical itself.* There are, occasionally, some apparent exceptions to the rule, when the expression can be really simplified by other processes. But the exceptions are only apparent, and result from the fact, that the expressions are not in proper form for the application of the rule. Sometimes a factor is under the second radical, which should be in the coefficient; sometimes one in the coefficient which should be under the radical. Thus,

$\sqrt{(6 \pm \sqrt{32})} = \sqrt{(-26 \pm \sqrt{32} + 32)}$, and the rule does not apply; but $\sqrt{32} = 4\sqrt{2}$, and substituting,

$$\sqrt{(6 \pm 4\sqrt{2})} = \sqrt{(4 \pm 4\sqrt{2} + 2)} = 2 \pm \sqrt{2}.$$

$\sqrt{(46 \pm 6\sqrt{5})} = \sqrt{(41 \pm 6\sqrt{5} + 5)}$; and the rule does not apply. But $6\sqrt{5} = 2\sqrt{45}$; substituting

$$\sqrt{(46 \pm 2\sqrt{45})} = \sqrt{(1 \pm 2\sqrt{45} + 45)} = 1 \pm \sqrt{45} = 1 \pm 3\sqrt{5}.$$

Of course there are many expressions which cannot be simplified; but when the expression can be reduced, the rule given is the shortest and the most direct for obtaining it in its simplest form.

## NOTE ON DERIVATIVES.

BY REV. THOMAS HILL.

1. To find by the aid of derivatives the development of $(a + x)^n$. Let us suppose

$$(1) \qquad (a + x)^n = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

the problem is to determine the values of $A$, $B$, $C$, &c.

Take the derivatives relatively to $x$, and also to $a$, denoting the latter by accenting the letters, that is, putting $A' = D_a A$, &c.

This gives

(2) $\quad n(a+x)^{n-1} = B + 2\,Cx + 3\,D\,x^2 + 4\,E\,x^3 + \&c.$

(3) $\quad n(a+x)^{n-1} = A' + B'\,x + C'\,x^2 + D'\,x^3 + \&c.$

As these are to be true for all values of $x$ we may put $x = 0$ by which (1) gives us

$$a^n = A,$$

while (2) and (3) give

$$n\,a^{n-1} = B = A'.$$

As the first members of (2) and (3) are equal, the second members are also, and will remain so, after subtracting the equal quantities $B$ and $A'$. This gives

$$2\,Cx + 3\,D\,x^2 + \&c. = B'\,x + C'\,x^2 + \&c.$$

Dividing by $x$ we have

$$2\,C + 3\,D\,x + \&c. = B' + C'\,x + \&c.$$

This by making $x = 0$ gives

$$2\,C = B';\ C = \tfrac{1}{2}\,B'.$$

A repetition of the process will give us the series,

$$\begin{aligned}
&A = a^n,\\
&B = A' = n\,a^{n-1},\\
&C = \tfrac{1}{2}\,B' = \tfrac{1}{2}\,n\,(n-1)\,a^{n-2},\\
&D = \tfrac{1}{3}\,C' = \tfrac{1}{3}.\,\tfrac{1}{2}\,n\,(n-1)\,(n-2)\,a^{n-3},\\
&E = \tfrac{1}{4}D' = \tfrac{1}{4}.\,\tfrac{1}{3}.\,\tfrac{1}{2}\,n\,(n-1)\,(n-2)\,(n-3)\,a^{n-4},
\end{aligned}$$

and so on.

These values of $A$, $B$, $C$, &c., being substituted in (1) give us

$$(a+x)^n = a^n + n\,a^{n-1}\,x + \frac{n\,(n-1)}{2}\,a^{n-2}\,x^2 + \&c.$$

2. To find, in derivatives of a function of $x$, the value of the function after $x$ has been increased by a definite amount.

Let us denote any function of $x$ by the expression $f\,(x)$, and the same function after adding $h$ to $x$ by the expression $f\,(x+h)$.

Let $h$ be divided into an infinite number of equal parts, either of which $= i$. Then it is manifest, since

$$\frac{f(x+i)-f(x)}{i}=Df(x),$$

that

$$(4) \qquad f(x+i)=f(x)+iDf(x).$$

And if in this, we put $x'$ for $x$,

$$f(x'+i)=f(x')+iDf(x').$$

Then putting $x'=x+i$, we have by (4)

$$f(x+2i)=f(x)+iDf(x)+iDf(x)+i^2D^2f(x),$$
$$=f(x)+2iDf(x)+i^2D^2f(x).$$

In like manner

$$f(x+3i)=f(x)+3iDf(x)+3i^2D^2f(x)+i^3D^3f(x).$$

And in general

$$f(x+ni)=f(x)+\frac{n}{1}Df(x)+\frac{n(n-1)i^2}{1.2}D^2f(x)+\&c.$$

And if we make $n$ infinite so that $n-1$, $n-2$, $n-3$, &c. are each equal to $n$, we have

$$f(x+ni)=f(x)+\frac{ni}{1}Df(x)+\frac{(ni)^2}{1.2}D^2f(x)+\frac{(ni)^3}{1.2.3}D^3f(x)+\&c.$$

Putting $ni=h$, that is, making $n$ the number of parts in $h$, gives

$$(5) \quad f(x+h)=f(x)+\frac{hDf(x)}{1}+\frac{h^2D^2f(x)}{1.2}+\frac{h^3D^3f(x)}{1.2.3}+\&c.$$

Making $x=0$, gives us

$$(6) \quad f(h)=f(0)+\frac{hDf(0)}{1}+\frac{h^2D^2f(0)}{1.2}+\frac{h^3D^3f(0)}{1.2.3}+\&c.$$

3. Equation (5) is called Taylor's theorem and equation (6) MacLaurin's theorem. Their great practical value is apparent at a single glance, and their metaphysical value is no less deserving of notice, as showing how each point of a curve contains in itself the law of the whole line. No student of ordinary algebra need find any difficulty in mastering these beautiful theorems as I have now presented them.

## ON THE RELATION BETWEEN THE STATES OF MINIMUM AND EQUILIBRIUM.

BY JOHN PATERSON.

[*Continued from page* 14.]

1. Three equal translatory forces applied at the points $A, B, C$ (fig. 4), and acting either all towards or all from the point $O$ through the intervention of the rigid lines $A\,O, B\,O, C\,O$, every pair of which makes an angle of 120° at $O$, will equilibrate on that point. For, on $O$ describe a circle with any radius, and draw a line $L\,L'$ in any direction through the centre $O$: the algebraical sum of the cosines $O\,c' + O\,c'' + O\,c'''$ formed by the unit measures $O\,a$, $O\,b$, $O\,c$, of the given forces, with the line $L\,L'$, is zero; the sum of the components of two of the unit forces is equal in magnitude and opposed in direction to the component of the third force in the same direction, and the direction of each force divides equally the angle formed by the two others.

Fig 4.

The lines $O\,A$, $O\,B$, $O\,C$, may be prolonged indefinitely, and the points $A, B, C$ taken anywhere on each: arcs to contain each an angle of 120°, described on $A\,B$, $B\,C$, $C\,A$ as chords, will always intersect in $O$, which is always, therefore, the point of aggregate minimum distances, so long as neither of the points passes over the point $O$.

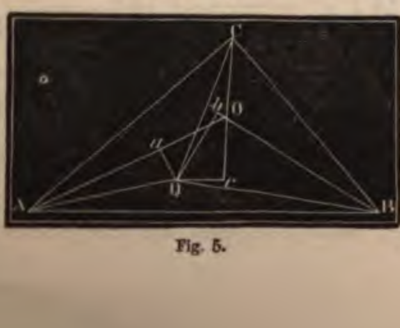


Fig. 5.

2. To show that $O$ is the point of minimum distance, SIMPSON directs to assume some other point $Q$ (fig. 5), and draw $Q\,a$, $Q\,b$, $Q\,c$ respectively perpendicular to $A\,O, B\,O, C\,O$; the sum of the prolongations $O\,b + O\,c$ is equal

to the curtailment $O a$, and consequently

$$A a + B b + C c = A O + B O + C O.$$

Draw the three hypothenuses $A Q$, $B Q$, $C Q$ to form the right angled triangles $A a Q$, $B b Q$, $C c Q$, and

$$A Q + B Q + C Q > A O + B O + C O.$$

3. The point of aggregate minimum distance having thus been found by geometry, the final solution of the problem appertains to ordinary trigonometry. Making $A O = x$, $B O = y$, $C O = z$, the triangles $A O B$, $B O C$, $C O A$ give the equations

$$c^2 = x^2 + y^2 + x y,\ a^2 = y^2 + z^2 + y z,\ b^2 = z^2 + x^2 + z x,$$

because $\cos 120° = -\frac{1}{2}$. The elimination being duly performed, the results are

$$x = \frac{c^2 - b^2 + a^2 + \sqrt{[\frac{2}{3}(a^2 b^2 + b^2 c^2 + c^2 a^2) - \frac{1}{3}(a^4 + b^4 + c^4)]}}{(2(a^2 + b^2 + c^2) + 6\sqrt{[\frac{2}{3}(a^2 b^2 + b^2 c^2 + c^2 a^2) - \frac{1}{3}(a^4 + b^4 + c^4)]})^{\frac{1}{2}}},$$

$$y = \frac{b^2 - a^2 + c^2 + \sqrt{[\frac{2}{3}(a^2 b^2 + b^2 c^2 + c^2 a^2) - \frac{1}{3}(b^4 + a^4 + c^4)]}}{(2(a^2 + b^2 + c^2) + 6\sqrt{[\frac{2}{3}(a^2 b^2 + b^2 c^2 + c^2 a^2) - \frac{1}{3}(a^4 + b^4 + c^4)]})^{\frac{1}{2}}},$$

$$z = \frac{a^2 - c^2 + b^2 + \sqrt{[\frac{2}{3}(a^2 b^2 + b^2 c^2 + c^2 a^2) - \frac{1}{3}(a^4 + b^4 + c^4)]}}{(2(a^2 + b^2 + c^2) + 6\sqrt{[\frac{2}{3}(a^2 b^2 + b^2 c^2 + c^2 a^2) - \frac{1}{3}(a^4 + b^4 + c^4)]})^{\frac{1}{2}}}.$$

This method of finding $x, y, z$ is given by Mr. Perkins, in his Treatise on Trigonometry; and the elimination is exhibited at length in the Treatise on Algebra, by the same author.

Another method, but more circuitous and operose, is followed by some writers, which consists in equating the sum of the expressions for the areas of the included triangles $A O B$, $B O C$, $C O A$, to the area of the including triangle $A B C$ expressed in terms of the sides $a, b, c$, and then eliminating the unknown quantities by artifice as usual. The equation thus formed is

$$\frac{x y + y z + z x}{4}\sqrt{3} = (S(S - a)(S - b)(S - c))^{\frac{1}{2}} = M^2,$$

when calculated, because $\sin 120° = \frac{\sqrt{3}}{2}$, and $S$ is put for the semi-

perimeter of the triangle; and the results of the elimination are expressed by

$$x=\frac{1}{R}\left(\frac{b^2+c^2-a^2}{2}+\frac{2M^2}{\sqrt{3}}\right),$$
$$y=\frac{1}{R}\left(\frac{a^2+c^2-b^2}{2}+\frac{2M^2}{\sqrt{3}}\right),$$
$$z=\frac{1}{R}\left(\frac{a^2+b^2-c^2}{2}+\frac{2M^2}{\sqrt{3}}\right),$$

where $R$ is put for

$$\sqrt{\left(\frac{a^2+b^2+c^2}{2}+2M^2\sqrt{3}\right)}.$$

4. On the occasion of a railroad company in the western part of this State (New York) occupying three stations forming a triangle with sides measuring 60, 70 and 90 miles, and desiring to locate a depot of stores for the mutual accommodation of the stations at the point of minimum travel, this noted and somewhat difficult problem has recently been completely re-solved by a practical arithmetician, without calling in the aid of the higher geometry, and apparently entirely unaware that the question was extant in history. When it is brought to mind that the solution taxed the ingenuity of one of Italy's ablest geometers, it will reflect much credit upon the mathematical ability of the calculator who has thus achieved a new solution by the principles of arithmetic alone; while at the same time it adds another to the already numerous instances, so frequently happening in our country, of an almost useless expenditure of time and effort at the discovery of things that were brought to light centuries ago. An acquaintance with the labors of our predecessors is a useful introduction to the prosecution of new discovery; and mathematicians would do well sometimes to apply a phrase of SHAKSPEARE'S, "and look to the past, as well as to the future."

The minimum distances in the present example, calculated from

the formulæ of Mr. PERKINS, and agreeing to the sixth decimal with the values reported to the railroad company, are

$$x = 57.0841787$$
$$y = 46.6646085$$
$$z = 21.0164356$$
$$124.7652228 \text{ miles} = AO + BO + CO,$$

the aggregate minimum distance.

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## OVALS AND THREE-CENTRE ARCHES.

BY J. B. HENCK.

*Application to certain Special Cases.*

IN applying formula (1), page 26, we may make various suppositions, such as that the angle $BED$ (Fig. 2) shall be 60°; that the first radius shall be a certain part of $a$; that $a$ and $b$ shall have a certain ratio to each other, &c. These suppositions will introduce additional equations which are to be combined with (1).

Fig 2.

*Case* I. When the angle $BED$ (Fig. 2) is to be 60°. This gives the angles $FEG$ and $EGF$ each $= 60°$, and consequently $EF = 2\,EC$, or

$$r_1 - r = 2\,(a - r),$$
$$\therefore\ r_1 = 2\,a - r.$$

Substituting this value of $r_1$ in (1) we have

$$2\,ar + 4\,ab - 2\,br - 4\,ar + 2\,r^2 = a^2 + b^2,$$
$$\therefore\ r^2 - (a + b)\,r = \tfrac{1}{2}\,(a^2 + b^2 - 4\,ab),$$
$$\therefore\ r = \tfrac{1}{2}\,(a + b) \pm \tfrac{1}{2}\,(a - b)\,\sqrt{3},$$

$$\therefore\ r = a - \tfrac{1}{2}(a-b) - \tfrac{1}{2}(a-b)\sqrt{3},^{*}$$

$$\therefore\ r = a - \tfrac{1}{2}(a-b)(1+\sqrt{3}),$$

and $$r_1 = 2a - r = a + \tfrac{1}{2}(a-b)(1+\sqrt{3}).$$

The construction of this case is usually given as follows: With centre $C$ and radius $a$ draw the quadrant $BHI$. Make $BH = HC = a$. Join $HI$. Through $A$ draw $AD$ parallel to $HI$, meeting $BH$ in $D$. Through $D$ draw $DEF$ parallel to $HC$, cutting $BC$ in $E$, and $AC$ produced in $F$. $E$ and $F$ are two of the centres.

Another construction, derived directly from the value of $r$ just found, may be used. On $HC$ make $KC = AI = a - b$. Through $K$ draw $KE$ parallel to $BI$. Through $E$ draw $DEF$ parallel to $HC$, cutting $AC$ produced in $F$. $E$ and $F$ are two of the centres. By means of triangular rulers, having angles of 60° and 45°, this construction becomes remarkably simple. The correctness of this construction is evident. For if the perpendicular $KL$ be dropped on $EC$, we have $LC = KC \cos 60° = \frac{1}{2}(a-b)$, and $EL = KL = KC \sin 60° = \frac{1}{2}(a-b)\sqrt{3}$ $\therefore$ $EC = LC + EL = \frac{1}{2}(a-b)(1+\sqrt{3})$, and $BE = BC - EC = a - \frac{1}{2}(a-b)(1+\sqrt{3})$.

In a work in common use in this country, the centre $E$ is obtained by laying off $EC = \frac{4}{3}(a-b)$. We have seen that it should be $\frac{1}{2}(1+\sqrt{3})(a-b) = 1.366\,(a-b)$ nearly. The error is too slight to be readily detected on a small figure on paper, but should an engineer lay out a large reservoir, for instance, by this rule (a misfortune that has actually happened), he would learn the value of knowing how to compute the radii accurately.

Fig. 3.

*Case* II. When the angle $BED$ (Fig. 3) is to be 45°. Here the angles

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lower sign alone of the double sign is to be used, since we know that $r$ an $b$, and, therefore, less than $\frac{1}{2}(a+b)$.

$FEC$ and $EFC$ are each $= 45°$, and consequently $CF = EC$ or

$$r_1 - b = a - r,$$

$$\therefore\ r_1 = a + b - r.$$

Substituting this value in (2), page 26, we have

$$a + b - r = a + \frac{(a-b)^2}{2(b-r)},$$

$$\therefore\ (b-r)^2 = \tfrac{1}{2}(a-b)^2,$$

$$\therefore\ b - r = \tfrac{1}{2}(a-b)\sqrt{2},$$

$$\therefore\ r = b - \tfrac{1}{2}(a-b)\sqrt{2},$$

and

$$\therefore\ r_1 = a + b - r = a + \tfrac{1}{2}(a-b)\sqrt{2}.$$

The construction is as follows: From $C$ lay off $CG$ and $CH$ each equal to $a - b$. Join $GH$, and from $C$ draw $CI$ perpendicular to $GH$. Then $GI = \frac{1}{2}(a-b)\sqrt{2}$, and as $BG = b$, lay off $GE = GI$, and $E$ will be one centre. The other centres $F$, $K$, and $L$ are obtained by making $CF$, $CK$, and $CL$ each equal to $CE$.

*Case* III. To draw an oval, so that the two radii may be as nearly equal as possible, that is, that $r_1 - r$ shall be a minimum. By subtracting $r$ from both sides of (2), we have

$$r_1 - r = a + \frac{(a-b)^2}{2b - 2r} - r.$$

Differentiating, we have

$$D_r(r_1 - r) = \frac{2(a-b)^2}{4(b-r)^2} - 1 = 0,$$

$$\therefore\ (b-r)^2 = \tfrac{1}{2}(a-b)^2,$$

$$\therefore\ b - r = \tfrac{1}{2}(a-b)\sqrt{2},$$

$$\therefore\ r = b - \tfrac{1}{2}(a-b)\sqrt{2}.$$

This is the same value of $r$ as that found in Case II.; so that to make the difference of the radii a minimum, the angle $BED$ must be 45°, and the construction is the same as that of Case II.

*Case* IV. When $a : b = 3 : 2$, and $r$ is taken equal to $\frac{1}{2}a$. Here (2) gives

$$r_1 = a + \frac{(a - \frac{2}{3}a)^2}{\frac{4}{3}a - a} = a + \frac{(\frac{1}{3}a)^2}{\frac{1}{3}a} = \tfrac{4}{3}a = 2b.$$


Fig. 4.

The construction of this case is extremely easy, and merits attention when the ratio of $a$ to $b$ can be assumed as above. The centres $E$ and $G$ (Fig. 4) bisect $B\,C$ and $C\,H$, and the other two centres $A$ and $F$ are at the extremities of the diameter $A\,F$.


Fig. 5.

*Case* V. When $D\,F$, the line passing through both centres (Fig. 5), is to be perpendicular to the whole chord $A\,B$. Here the triangle $E\,C\,F$ becomes similar to $A\,B\,C$, and we have

$$r_1 - b : a - r = a : b,$$
$$\therefore b\,r_1 - b^2 = a^2 - a\,r,$$
$$\therefore r_1 = \frac{a^2 + b^2 - a\,r}{b}.$$

If in this value of $r_1$ and in the first value given in (2) we put $a^2 + b^2 = c^2$, and then put the two values equal to each other, we have
$$\frac{c^2 - a\,r}{b} = \frac{c^2 - 2\,a\,r}{2\,b - 2\,r},$$
$$\therefore 2\,b\,c^2 - 2\,a\,b\,r - 2\,c^2 r + 2\,a\,r^2 = b\,c^2 - 2\,a\,b\,r,$$
$$\therefore r^2 - \frac{c^2 r}{a} = -\frac{b\,c^2}{2\,a},$$
$$\therefore r = \frac{c^2}{2\,a} \pm \sqrt{\frac{c^4 - 2\,a\,b\,c^2}{4\,a^2}},$$
$$\therefore r = \frac{c^2}{2\,a} \pm \frac{c}{2\,a}\sqrt{c^2 - 2\,a\,b},$$
$$\therefore r = \frac{c^2}{2\,a} \pm \frac{c}{2\,a}(a - b),$$
$$\therefore r = \frac{c\,(c - a + b)}{2\,a},$$
$$\therefore r_1 = \frac{c^2 - \frac{1}{2}\,c\,(c - a + b)}{b} = \frac{c\,(c + a - b)}{2\,b}.$$

These values give rise to a simple construction often given. On

$AB$ take $AH = a - b$, and bisect $BH$ by the perpendicular $DF$, cutting $BC$ in $E$, and $AC$, produced when necessary, in $F$. $E$ and $F$ are the centres. For the similar triangles $EBG$ and $ABC$ give

$$BE : BG = AB : BC,$$

or

$$r : \tfrac{1}{2}(c - a + b) = c : a,$$

$$\therefore r = \frac{c(c - a + b)}{2a};$$

and the similar triangles $AFG$ and $ABC$ give

$$AF : AG = AB : AC,$$

$$r_1 : \tfrac{1}{2}(c + a - b) = c : b,$$

$$\therefore r_1 = \frac{c(c + a - b)}{2b}.$$

## ON THE DIVISIBILITY OF NUMBERS.

BY E. B. ELLIOTT.

Any whole number, $N$, may take the usual decimal form

(1)
$$N = 10^p n_p + 10^{p-1} n_{p-1} + \ldots 10^2 n_2 + 10^1 n_1 + n_0,$$
$$= \Sigma_p 10^p n_p,$$

in which $n_p$, $n_{p-1}$, &c., are the digits in the order of their places, $n_0$ being the unit digit, $n_1$ the digit in the place of tens, and so on; $p+_1$ denotes the number of digits, and $\Sigma_p$ the sum taken relatively to $p$.

We wish a simple rule for determining by inspection when the number, $N$, is an exact multiple of a given integral divisor, $D$; and when not a multiple, for determining the number of units in the excess of $N$ over the largest multiple of the divisor, which may be contained in it.

Any power $p$ of 10 is equal to some multiple $M$ of the divisor $D$, plus a remainder $r_p$; that is

(2)
$$10^p = M\,D + r_p,$$

in which $M$ may be zero, or any integer, whatever, whether positive or negative. The numerical value of the remainder $r_p$ may be essentially either positive or negative.

By substituting this value of $10^p$ in (1), it becomes

$$
\begin{aligned}
N &= \Sigma_p (M\,D + r_p)\, n_p, \\
&= \Sigma_p n_p M\,D + \Sigma_p n_p r_p, \\
&= \Sigma_p n_p M\,D + (n_p r_p + n_{p-1} r_{p-1} + \ldots n_2 r_2 + n_1 r_1 + n_0 r_0).
\end{aligned}
$$

It is manifest that whenever the quantity in the parenthesis is a multiple of the divisor $D$, the number $N$ will be also.

It will commonly be desirable to have the numerical values of the remainders $r_p$, $r_{p-1}$, $r_{p-2}$, &c., the smallest integers possible, whether positive or negative. A convenient general method, for finding the numerical values of these remainders, will be to divide unity, followed by an indefinite number of zeros, by the given divisor $D$. The successive remainders will be the ones sought.

Example. Required, the numerical values of the remainders, $r_p$, $r_{p-1}$, $r_{p-2}$, &c., peculiar to the divisor seven.

$$
\begin{array}{r|llllllllll l}
7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
\hline
 & 0 & 1 & 4 & 3 & \bar{1} & \bar{4} & \bar{3} & 1 & 4 & \cdots & \text{Quotient.} \\
\hline
 & 1 & 3 & 2 & \bar{1} & \bar{3} & \bar{2} & 1 & 3 & 2 & \cdots & \text{Remainder.} \\
 & r_0 & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & &
\end{array}
$$

Negative quotients and remainders have the sign minus written over them. In the usual form of division the quotients and remainders will be positive.

$$
\begin{array}{r|llllllllll l}
7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
\hline
 & 0 & 1 & 4 & 2 & 8 & 5 & 7 & 1 & 4 & \cdots & \text{Quotient.} \\
\hline
 & 1 & 3 & 2 & 6 & 4 & 5 & 1 & 3 & 2 & \cdots & \text{Remainder.} \\
 & r_0 & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & &
\end{array}
$$

The number 454526 is not a multiple of seven, for the sum of once the unit digit $(1 \times 6)$, three times the next $(3 \times 2 = 6)$, twice the next $(2 \times 5 = 10)$, minus once the next $(-1 \times 4 = -4)$, minus three times the next $(-3 \times 5 = -15)$, minus twice

the next ($-2 \times 4 = -8$), is minus five ($6 + 6 + 10 - 4 - 15 - 8 = -5 = -1 \times 7 + 2$), which is not a multiple of seven; but the excess of minus five over the multiple of seven next smaller, that is, minus once seven, is two. Hence the excess of the number 454526 over the greatest contained multiple of seven is two.

For the negative values of the remainders ($-1, -3, -2$) may be substituted the positive values (6, 4, 5) obtained by adding seven to each negative value. But the repetend, in this case, would be one of six distinct places, namely (1, 3, 2, 6, 4, 5), instead of the simpler, the more easily remembered, and the more readily applied one of but three different places, namely ($1, 3, 2, -1, -3, -2$), the successive series being alternately positive and negative. The following table gives values of the remainders corresponding to a few of the practically more important divisors; and it may readily be continued for any other divisors whatever.

| *Divisors.* | *Remainders.* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *D.* | $r_0,$ | $r_1,$ | $r_2,$ | $r_3,$ | $r_4,$ | $r_5,$ | $r_6,$ | $r_7,$ | $r_8,$ | .... |
| 3. | 1, | 1, | 1, | 1, | 1, | 1, | 1, | 1, | 1, | .... |
| 7. | 1, | 3, | 2, | —1, | —3, | —2, | 1, | 3, | 2, | .... |
| | or 1, | 3, | 2, | 6, | 4, | 5, | 1, | 3, | 2, | .... |
| 8. | 1, | 2, | 4, | 0, | 0, | 0, | 0, | 0, | 0, | .... |
| 9. | 1, | 1, | 1, | 1, | 1, | 1, | 1, | 1, | 1, | .... |
| 11. | 1, | —1, | 1, | —1, | 1, | —1, | 1, | —1, | 1, | .... |
| | or 1, | 10, | 1, | 10, | 1, | 10, | 1, | 10, | 1, | .... |
| 13. | 1, | —3, | —4, | —1, | 3, | 4, | 1, | —3, | —4, | .... |
| | or 1, | 10, | 9, | 12, | 3, | 4, | 1, | 10, | 9, | .... |
| 17. | 1, | —7, | —2, | —3, | 4, | 6, | —8, | 5, | —1, | .... |
| | or 1, | 10, | 15, | 14, | 4, | 6, | 9, | 5, | 16, | .... |
| 99. | 1, | 10, | 1, | 10, | 1, | 10, | 1, | 10, | 1, | .... |
| 101. | 1, | 10, | —1, | —10, | 1, | 10, | —1, | —10, | 1, | .... |
| | or 1, | 10, | 100, | 91, | 1, | 10, | 100, | 91, | 1, | .... |

The remainders peculiar to the number *seventeen* may take the form either of a positive repetend of sixteen places, or of repetends alter-

nately positive and negative of but eight places each, the complete series being

1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, ....

or 1, —7, —2, —3, 4, 6, —8, 5, —1, 7, 2, 3, —4, —6, 8, —5, ....

or 1, 10, —(2, 3, —4) + 3 (2, 3, —4), —1, —10, + (2, 3, —4,) —3 (2, 3, —4), ....

The above general method for finding the numerical values of the remainders $(r_0, r_1, r_2, r_3, \ldots.)$ peculiar to any given divisor, accompanied by their numerical values for all integral divisors between 1 and 30, was given by the present writer in the year 1846 to classes of pupils at Lyons, in the State of New York.

In the fourth volume of the "Mémoires de la Société Impériale des Sciences Naturelles de Cherbourg," published in 1856, appears a paper entitled "Caractères de Divisibilité des Nombres Entires, par M. de Lapparent," in which it is stated that "the celebrated shepherd computer of Touraine, Henri Mondeux, in his last passage to Cherbourg in February, 1856, had offered for sale a pamphlet in which he indicated the characters of divisibility of numbers by the divisors comprised between 1 and 50. These very ingenious formulas were not accompanied by any theory."

"In fact, the former instructor of Mondeux, M. Jacobi, in the preface to the pamphlet in question, says, 'that his pupil had no knowledge of the principles upon which these formulas could be based, and that he himself avowed, in all humility, his ignorance of them.'" The object of the memoir of M. de Lapparent was to investigate these principles.

The properties of numbers which were indicated by Mondeux, so far as can be ascertained from the few examples given in the above-mentioned memoir, do not essentially vary from the theory and processes which we have suggested above, but differ widely from the very ingenious formulas of Lapparent.

Hereafter we purpose discussing the subject from another point of view, and one which will embrace the theory of M. de Lapparent.

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### QUERY BY MAJOR J. G. BARNARD.

Is there any *general* proof that

(1) $$\varphi\,(y + x\sqrt{-1}) + \varphi\,(y - x\sqrt{-1}),$$

and

(2) $$\frac{1}{\sqrt{-1}}\{\varphi\,(y + x\sqrt{-1}) - \varphi\,(y + x\sqrt{-1})\}$$

are in all cases *real quantities*, $\varphi$ being any arbitrary function which is supposed to contain nothing imaginary in its form? Mr. Airy assumes it.

*Answer by the Editor.* Put $h = x\sqrt{-1}$, and develop $\varphi\,(y + h)$ and $\varphi\,(y - h)$ by Taylor's theorem. We get

(3) $$\varphi\,(y + h) = \varphi y + \varphi' y . \frac{h}{1} + \varphi'' y . \frac{h^2}{1.2} + \varphi''' y . \frac{h^3}{1.2.3} + \&\text{c.}$$

(4) $$\varphi\,(y - h) = \varphi y - \varphi' y . \frac{h}{1} + \varphi'' y . \frac{h}{1.2} - \varphi''' y . \frac{h^3}{1.2.3} + \&\text{c.}$$

Hence, by addition,

$$\varphi\,(y + h) + \varphi\,(y - h) = 2\,\varphi y + 2\,\varphi'' y . \frac{h^2}{1.2} + 2\,\varphi'''' y . \frac{h^4}{1.2.3.4} + \&\text{c.}$$

which contains only even powers of $h$, and is therefore always real, so long as $\varphi y$ is real. If instead of adding, we take the difference of these developments, the even powers of $h$ disappear, and the result,

$$\varphi\,(y + h) - \varphi\,(y - h) = 2\,\varphi' y . \frac{h}{1} + 2\,\varphi''' y . \frac{h^3}{1.2.3}$$
$$+ 2\,\varphi^{v} y . \frac{h^5}{1.2.3.4.5} + \&\text{c.},$$

contains only odd powers; and therefore every term contains $\sqrt{-1}$

as a factor which may be divided out, so that (2) is always real for real values of $\varphi y$.

If we denote the real parts of (3) and (4) by $P$, and the imaginary parts by $Q\sqrt{-1}$, they become

$$(5)\qquad \varphi(y+h) = P + Q\sqrt{-1},$$
$$(6)\qquad \varphi(y-h) = P - Q\sqrt{-1}.$$

Hence, if $y + x\sqrt{-1}$, and $y - x\sqrt{-1}$ are imaginary conjugates, like functions of these quantities are also.

The more general expression

$$\psi(u + v\sqrt{-1})\,\varphi(y + x\sqrt{-1}) \pm \psi(u - v\sqrt{-1})\,\varphi(y - x\sqrt{-1})$$

may, by (5) and (6), be written in the form

$$(P + Q\sqrt{-1})(P' + Q'\sqrt{-1}) \pm (P - Q\sqrt{-1})(P' - Q'\sqrt{-1}),$$

which is readily seen to be real or imaginary according as we use the sign plus or minus. Of course, it has been assumed in what precedes, that real functions of imaginary quantities can be developed by the same methods, and into the same forms, as real quantities; also, that reasoning based upon possible development is conclusive.

## THE PHILOSOPHY OF ALGEBRAIC SIGNS.

BY PROF. W. D. HENKLE.*

ONE of the most important things for the mathematical student to understand is the true nature of the signs + and —. I propose to explain their use solely upon the doctrine that they are signs of operations whose effects are directly opposite. When + denotes addition, — denotes subtraction; when + denotes multiplication, — denotes division; when + denotes subtraction, — denotes addition. It must be remembered that a different idea is attached to *a*

* Editor of the Indiana School Journal, Indianapolis, Indiana.

and $+ a$, the former merely indicating a quantity, and the latter a specific operation. To subtract $b$ from $a$ is merely a simple arithmetical operation, and the operation may be indicated by $a - b$, but to subtract $+ b$ from $a$ involves an additional idea, for it means the subtraction from $a$ of the specific operation indicated by $+ b$. Let us assume that the quantity $a$ is the result obtained by adding $b$ to $R$, that is, $a = R + b$; now if the operation $+ b$ in $R + b$ is subtracted, omitted, or neglected, we have merely the quantity $R$. It is evident then, that $R$ is the result obtained by subtracting the operation $+ b$ from $a$, that is, $R = a - (+ b)$. We can change $a - (+ b)$ into an arithmetical equivalent in the following manner: subtract $b$ (not $+ b$) from $a$, and to the result add $b$ (not $+ b$), and we have $a - b + b$, an expression which, considered in reference to the result obtained by performing the operations indicated, is equivalent to $a$; hence $a - (+ b)$ is equivalent to the result obtained by omitting in $a - b + b$ the operation $+ b$. This establishes the fact that $a - (+ b) = a - b$, or that $+ b$ subtracted from $a$ gives the same result as $b$ subtracted from $a$. If we desire to subtract $- b$ from $a$, we write $a + b - b$ for $a$, and then omit the operation $- b$, showing that the result obtained by subtracting $- b$ from $a$ is the same as when $b$ is added to $a$.

Let us now subtract $b - c$ from $a$. Considering $b$ as a quantity, not as an operation, we have $a - b$ for the subtraction of $b$ from $a$, and the operation $- c$ subtracted from $a - b + c - c$, or $a - b$, gives $a - b + c$.

Let us subtract 5 from 3. It is evident that this is impossible, for the less does not contain the greater; but 3 can be subtracted from 5, thus leaving 2 yet demanding the operation of subtraction. This fact is represented by $- 2$, that is, $3 - 5 = - 2$.

In Analytical Geometry, we have positive and negative abscissas, and ordinates. For instance, $x = + 3$, and $y = - 3$, means the

point reached by going on the axis of abscissas 3 to the right of the origin, and then going 3 below the axis of abscissas on a line parallel to the axis of ordinates. Of course it is purely conventional that $+$ indicates motion to the right, or above, and $-$ motion to the left, or below.

A *positive* quotient in division indicates the number of times that the divisor must be *subtracted* from the dividend to obtain zero; and a *negative* quotient, the number of times that the divisor must be *added* to the dividend to obtain zero.

In the case of exponents we have $+$ indicating multiplication and $-$ division. We remark, first, that $a$ denotes a quantity, but $a^1$ denotes the multiplication of 1 by $a$; also $a^2$ denotes the multiplication of 1 by $a$ twice; that is, is the product of 1, $a$, and $a$. If we now agree to extend this simple arithmetical idea by saying that $a^{+3}$ shall be the same as $a^3$, we have the means of introducing negative exponents. Thus if $a^{+3}$ denotes 1 *multiplied* by $a$ three times, $a^{-3}$ might consistently denote 1 divided by $a$ three times. If the word *functionize* meant *multiply* or *divide*, we might define the exponent of a quantity to represent the number of times unity is to be functionized by that quantity. We see now how $a^0$ equals 1, for it means that *one* is to be functionized no time by $a$, and therefore it must still be 1.

It might be asked why it would not be philosophical to have a positive exponent denote involution, and a negative one, evolution; that is, why should not $a^{-3}$ denote the cube root of $a$, if $a^{+3}$ denotes the third power of $a$? I answer that such a representation would prevent the use of fractional exponents, and limit the rule in reference to the addition and subtraction of exponents in multiplication and division.

NOTE. — Perhaps it may not be considered proper to speak of subtracting $+b$ from $a$, unless $a$ is changed to $+a$. This change is evidently legitimate, whenever necessary.

## EXTENSION OF THE PRISMOIDAL FORMULA.

BY CHAUNCEY WRIGHT.

We have shown that the Prismoidal Formula is an exact definite integral for algebraic expressions of positive integer powers not exceeding the third degree, and that it is sufficiently accurate in all cases where the fourth order of differences may be neglected. We propose now to determine similar formulas, which shall include higher orders of differences.

If we represent by

$$f(x),\ f(x+h_1),\ f(x-h_1),\ f(x+h_2),\ f(x-h_2),\ \&c.$$

a series of sections made perpendicular to the axis of $X$, through a solid or plane figure, and at the distances $x, x+h_1, x-h_1$, &c. from the origin of $x$, then the expression

$$(A)\quad \begin{aligned} &h_n\,[a_n f(x-h_n)+\ldots+a_2 f(x-h_2)+a_1 f(x-h_1) \\ &+afx+a_1 f(x+h_1)+a_2 f(x+h_2)+\ldots+a_n f(x+h_n)], \end{aligned}$$

which contains $2n+1$ terms, $n+1$ coefficients $a, a_1, a_2$, &c., and $n$ values $h_1, h_2$, &c., mây be made to represent the contents of any portion of the solid or plane figure for which $f(x)$ is an algebraic expression of positive integer powers not exceeding the $(4n-1)$th degree.

Formula $(A)$ may be written thus:—

$$h_n\,\Sigma_n a_n\,[f(x+h_n)+f(x-h_n)]+h_n\,a\,f(x),$$

and developed by Taylor's Theorem, it becomes

$$\left.\begin{aligned} &2h_n\,\Sigma_n a_n\left[f(x)+\frac{h_n^2}{1.2}\overset{\text{II}}{f}(x)+\frac{h_n^4}{2.3.4}\overset{\text{IV}}{f}(x)+\&c.\right] \\ &\qquad\qquad\qquad +h_n\,a\,f(x) \end{aligned}\right\}=$$

$$(B)\left\{\begin{aligned} &2h_n\left[(\tfrac{1}{2}a+a_1+a_2+\&c.)f(x)+\tfrac{1}{2}(a_1h_1^2+a_2h_2^2+\&c.)\overset{\text{II}}{f}(x)\right. \\ &\qquad\qquad\qquad\left.+\frac{1}{2.3.4}(a_1h_1^4+a_2h_2^4+\&c.)\overset{\text{IV}}{f}(x)\right]. \end{aligned}\right.$$

Now, the definite integral

$$\int \left[f(x+h_n) - f(x-h_n)\right] dx$$

becomes, when developed,

$$(C) \qquad \begin{aligned} & 2\int \left[h_n \overset{\mathrm{I}}{f}(x) + \frac{h_n^3}{2.3} \cdot \overset{\mathrm{III}}{f}(x) + \frac{h_n^5}{2.3.4.5} \overset{\mathrm{V}}{f}(x) + \&c.\right] dx = \\ & 2 h_n \left[f(x) + \frac{h_n^2}{2.3} \overset{\mathrm{II}}{f}(x) + \frac{h_n^4}{2.3.4.5} \overset{\mathrm{IV}}{f}(x) + \&c.\right], \end{aligned}$$

and, if we make the corresponding terms of $(B)$ and $(C)$ equal, we have the following equations

$$\begin{aligned} 1 &= \tfrac{1}{2} a + a_1 + a_2 + \ldots + a_n &&= \Sigma_n (a_n) + \tfrac{1}{2} a, \\ \frac{h_n^2}{3} &= \quad a_1 h_1^2 + a_2 h_2^2 + \ldots + a_n h_n^2 &&= \Sigma_n (a_n h_n^2), \\ \frac{h_n^4}{5} &= \quad a_1 h_1^4 + a_2 h_2^4 + \ldots + a_n h_n^4 &&= \Sigma_n (a_n h_n^4), \\ \frac{h_n^6}{7} &= \quad a_1 h_1^6 + a_2 h_2^6 + \ldots + a_n h_n^6 &&= \Sigma_n (a_n h_n^6), \end{aligned}$$

and so on.

By putting $r_1 = \frac{h_1}{h_n}$, $r_2 = \frac{h_2}{h_n}$, &c., these equations become

$$(D) \qquad \begin{aligned} 1 &= \tfrac{1}{2} a + a_1 + a_2 + \ldots + a_n, \\ \tfrac{1}{3} &= \quad a_1 r_1^2 + a_2 r_2^2 + \ldots + a_n, \\ \tfrac{1}{5} &= \quad a_1 r_1^4 + a_2 r_2^4 + \ldots + a_n, \\ \tfrac{1}{7} &= \quad a_1 r_1^6 + a_2 r_2^6 + \ldots + a_n, \end{aligned}$$

and so on, in which $a$, $a_1$, $a_2$, &c., are the coefficients of the sections, and $r_1$, $r_2$, &c., the ratios of their distances from the middle section to the distance $h_n$ of the extreme sections from the middle.

If by $l$ we denote the whole distance between the extreme sections, then $h_n = \frac{l}{2}$.

The number of the equations $(D)$ which can be satisfied is equal to the number of undetermined quantities $a, a_1, a_2, \ldots a_n$; $r_1, r_2, \ldots r_{n-1}$, that is, to $(n+1) + (n-1) = 2n$. But the $(2n+1)$th

terms of the expressions $(B)$ and $(C)$ contain the coefficients of the $4n$th derivative of $f(x)$, which are therefore the first not included in the equations $(D)$; hence $(A)$ may be made accurate for $(4n-1)$ derivatives of $f(x)$. When the sections are made at equal intervals, the distances $h_1, h_2$, &c., and the ratios $r_1, r_2$, &c., are determined by their number, and $(A)$ can be made accurate only for as many derivatives of $f(x)$ as there are sections, that is, for $2n+1$. Since there are $2n$ equations $(D)$ the last will be of the form $\frac{1}{4n-1} = a_1 r_1^{4n-2} + a_2 r_2^{4n-2} \ldots + a_n$, and the error of $(A)$ for functions of two higher orders of derivatives, that is for functions which have $4n$, or $4n+1$ derivatives, is

$$\frac{2h_n^{4n+1}}{1.2.3\ldots 4n}\left(a_1 r_1^{4n} + a_2 r_2^{4n} + \ldots + a_n - \frac{1}{4n+1}\right) f^{[4n]} x.$$

EXAMPLES.

1. When there are only three sections, $n = 1$, and the equations $(D)$ become $1 = \frac{1}{2} a + a_1$ and $\frac{1}{3} = a_1$, hence $a = \frac{4}{3}$ and $(A)$ becomes $\frac{h_n}{3}\Big[f(x-h_1) + 4fx + f(x+h_1)\Big] = \frac{l}{6}\Big(B + 4B'' + B'''\Big)$, which is the common prismoidal formula.

2. When $n = 2$, there are five sections, and the equations $(D)$ become

$$\begin{aligned}
1 &= \tfrac{1}{2} a + a_1 + a_2, \\
\tfrac{1}{3} &= \qquad a_1 r_1^2 + a_2, \\
\tfrac{1}{5} &= \qquad a_1 r_1^4 + a_2, \\
\tfrac{1}{7} &= \qquad a_1 r_1^6 + a_2.
\end{aligned}$$

If these sections are made at equal intervals, then $r_1$ is determined, and only three of these equations can be satisfied; that is, only five orders of derivatives can be included by them. As the second and fourth sections will in this case bisect the intervals between the middle and the extreme sections, $r_1 = \frac{1}{2}$ and the equations

$$1 = \tfrac{1}{2} a + a_1 + a_2, \quad \tfrac{1}{3} = a_1 \tfrac{1}{4} + a_2, \quad \tfrac{1}{5} = a_1 \tfrac{1}{16} + a_2,$$

give $\qquad a = \frac{12}{45},\quad a_1 = \frac{32}{45},\quad a_2 = \frac{7}{45},$

so that the formula $(A)$ becomes

(1) $\qquad \frac{l}{90}(7B + 32B' + 12B'' + 32B''' + 7B^{\text{IV}}),$

and its error for functions of six or seven derivatives is

$$\frac{\overset{\text{VI}}{f}(x) \,.\, l^7}{1935360}.$$

If $r_1$ be taken so that the coefficient $a$ of the middle section shall disappear, then

$$r_1 = \sqrt{\tfrac{1}{5}},\quad a_1 = \tfrac{5}{6},\quad a_2 = \tfrac{1}{6},$$

and we obtain a formula of four sections,

(2) $\qquad \frac{l}{12}(B + 5B' + 5B'' + B'''),$

in which the interval between $B'$ and $B''$ is $\frac{l}{\sqrt{5}}$.

This formula is accurate for five derivatives, and its error for functions of six or seven derivatives is

$$\frac{\overset{\text{VI}}{f}(x) \,.\, l^7}{1512000}.$$

If $r_1$ is left indeterminate, then the solution of the four equations above will give

$$a = \tfrac{64}{90},\quad a_1 = \tfrac{49}{90},\quad a_2 = \tfrac{9}{90},\quad r_1 = \sqrt{\tfrac{3}{7}},$$

and we obtain the formula of five sections

(3) $\qquad \frac{l}{180}\left(9B + 49B' + 64B'' + 49B''' + 9B^{\text{IV}}\right),$

for which the sections $B'$ and $B'''$ are at the distance $\frac{l}{2}\sqrt{\frac{3}{7}}$ from the middle section $B''$. This formula is accurate for seven derivatives, and its error for functions of eight or nine derivatives is

$$\frac{\overset{\text{VIII}}{f}(x) \,.\, l^9}{237081600}.$$

This error and those of formulas (1) and (2) may be estimated in terms of the finite differences of $f(x)$ by the following transformations. If the length $l$ be divided by sections into so many parts $m$ that the corresponding differences of $f(x)$ of a higher order than the $i$th, may be neglected, then

$$\Delta^i f(x) = \left(\frac{l}{m}\right)^i f^i(x), \text{ or } l m^i \Delta^i f(x) = l^{i+1} f^i(x),$$

so that, for instance, the error of formula (3) may be written

$$\frac{l\, m^8 \Delta^8 f(x)}{237081600}.$$

Hence, if the number of parts $m$ into which $l$ is divided be about the eighth root of 237081600, or about 11, and if the corresponding value of $\Delta^8 f(x)$ be inappreciable, then formula (3) is sufficiently accurate. In the same way the accuracy of formulas (1) and (2) may be estimated in practice.

In the application of formulas (1), (2), and (3) to symmetrical figures, as for instance for determining the contents of casks, since the sections at equal distances from the middle section are equal, these formulas may be written thus: —

$$\frac{l}{45}(7B + 32B' + 6B''),$$

$$\frac{l}{6}(B + 5B'),$$

$$\frac{l}{90}(9B + 49B' + 32B'').$$

Cases in practice might arise in which sections at equal intervals could not be obtained. For such cases special formulas might be easily obtained from the equations ($D$).

In some future number of this journal we shall apply the formulas of the preceding discussion to a variety of problems in Engineering, Tonnage of Vessels, Cask Gauging, &c.

## SOLUTION OF PROF. PEIRCE'S PROBLEM.

BY WILLIAM WATSON.

"WHAT curve is represented by the equation

$$\sin \varepsilon = a \operatorname{Cos} (\log r + b),$$

in which $r$ denotes the radius vector, and $\varepsilon$ the angle between $r$ and the corresponding tangent?"

Since $\operatorname{Cos} \alpha = \frac{1}{2} (c^{\alpha} + c^{-\alpha})$, in which $c$ denotes the Naperian base, we have

$$(1) \qquad \sin \varepsilon = \frac{a}{2} (c^{\log r + b} + c^{-\log r - b}) = \frac{a}{2}\left(c^b r + \frac{1}{c^b r}\right).$$

Let $p$ denote the perpendicular dropped from the origin on the tangent; and since $p = r \sin \varepsilon$, (1) multiplied by $r$ becomes

$$r \sin \varepsilon = p = \frac{a}{2}\left(c^b r^2 + \frac{1}{c^c}\right). \quad \therefore D_r p = a c^b r.$$

But the radius of curvature, $\varrho = \frac{1}{2} D_p r^2 = r D_p r = \frac{1}{a c^b} =$ constant. Therefore the curve is the circumference of a circle.

NOTE. — This form of $\varrho = \frac{1}{2} D_p r^2$ is not new, but may be deduced as follows. Let $ds$ be an element of the curve, $\varphi$ and $\tau$ the angles, made by $r$ and the tangent, with the prime radius. Since $r \sin \varepsilon = p$, we find

$$\cos \varepsilon = \frac{\sqrt{r^2 - p^2}}{r}, \; d\varepsilon = \frac{d \sin \varepsilon}{\cos \varepsilon} = \frac{r\,dp - p\,dr}{r\sqrt{r^2 - p^2}}.$$

Also,

$$ds = \frac{dr}{\cos \varepsilon} = \frac{r\,dr}{\sqrt{r^2 - p^2}}; \; d\varphi = \frac{ds \sin \varepsilon}{r} = \frac{p\,dr}{r\sqrt{r^2 - p^2}}.$$

But $\tau = \varphi + \varepsilon \therefore d\tau = d\varphi + d\varepsilon = \frac{dp}{\sqrt{r^2 - p^2}}$; and substituting these values of $ds$ and $d\tau$ in the value of $\varrho = \frac{ds}{d\tau}$, we get

$$\varrho = \frac{r\,dr}{\sqrt{r^2 - p^2}} \cdot \frac{\sqrt{r^2 - p^2}}{dp} = \frac{r\,dr}{dp} = \frac{1}{2} D_p r^2.$$

Or thus. Let $r_\varepsilon$ be the radius vector of the evolute corresponding to $r$ of the curve; then from the geometry of the figure,

$$r_\varepsilon^2 = (\varrho - p)^2 + r^2 - p^2 = \varrho^2 - 2p\varrho + r^2. \quad \text{Differentiating } \varrho = \tfrac{1}{2} D_p r^2.$$

## ANOTHER SOLUTION.

BY CHAUNCEY WRIGHT.

The equation $\sin \varepsilon = a \operatorname{Cos} (\log r + b),$ (1)
may be changed to the exponential form

$$\sin \varepsilon = \frac{a}{2} (c^{\log r} c^b + c^{-\log r} c^{-b}) = \frac{a c^b r}{2} + \frac{a}{2 c^b r};$$

and thence

$$r \sin \varepsilon = \frac{a c^b r^2}{2} + \frac{a}{2 c^b}. \qquad (2)$$

If we assume a polar axis, and suppose a normal to be drawn from any point of the curve to this axis, and denote by $N$ the length of the normal, and by $l$ the portion of the axis cut off by the normal; then in the triangle formed by $r$, $l$, and $N$ the angle included by $r$ and $N$ is $\frac{1}{2}\pi - \varepsilon$, and by Trigonometry

$$l^2 = N^2 + r^2 - 2 N r \cos (\tfrac{1}{2}\pi - \varepsilon) = N^2 + r^2 - 2 N r \sin \varepsilon;$$

so that

$$r \sin \varepsilon = \frac{r^2}{2 N} + \frac{N^2 - l^2}{2 N}. \qquad (3)$$

Now, in this general equation such values of $N$ and $l$ may be substituted, as will satisfy equation (2) for any possible values of $a$ and $b$. If we take

$$N_1 = \frac{1}{a c^b} \text{ and } \frac{N_1^2 - l_1^2}{N_1} = \frac{a}{c^b} = a^2 N_1,$$

and thence

$$l_1 = \frac{\sqrt{1 - a^2}}{a c^b},$$

and substitute these values in (3) we obtain (2); but these values are constant; hence the curve is a circle for all values of $b$, and for all values of $a^2$ less than unity, and since equation (1) is impossible when $a$ is greater than unity, it is the equation of a circle for all possible values of its constants. The radius of the circle is $\frac{1}{a c^b}$, and the origin of $r$ is at the distance $\frac{\sqrt{1 - a^2}}{a c^b}$ from the centre.

## A THIRD SOLUTION.

BY GEORGE B. VOSE.

PUT $\log r + b = u$; whence $D_u r = r$ (1). Let $\varphi$ denote the angle which the radius vector makes with a given line, and we have the relation $D_r \varphi = \frac{\tan \varepsilon}{r}$. (2). Multiplying together (1) and (2) we obtain

$$D_u \varphi = \tan \varepsilon = \frac{a \operatorname{Cos} u}{\sqrt{1 - a^2 \operatorname{Cos}^2 u}},$$

the integral of which is

$$\sin(\varphi + \varphi_0) = \frac{a \operatorname{Sin} u}{\sqrt{1 - a^2}} = \frac{a}{2\sqrt{1 - a^2}}\left(c^b r - \frac{1}{c^b r}\right),$$

in which $\varphi_0$ is the constant. Passing from polar to rectangular coördinates by means of the formulas $x = r \cos(\varphi + \varphi_0)$, and $y = r \sin(\varphi + \varphi_0)$, we finally obtain

$$c^b y = \frac{a}{\sqrt{1 - a^2}}\left(c^{2b}(x^2 + y^2) - 1\right),$$

which is the equation of a circle.

## PROBLEMS, BY PROF. PEIRCE.

INVESTIGATE the number of real roots in each of the following equations:—

I. $a^\theta + b^\theta + c = 0,$

II. $\sin^m \theta + a \sin^n \theta + b\,\theta + c = 0,$

III. $a^\theta + b\,\theta^n + c = 0.$

IV. Find the least sphere which shall contain a given system of points. This problem is an extension of the following:—

"It is required to find the least circle which shall contain a given system of points." — J. J. SYLVESTER, *The Quarterly Journal of Pure and Applied Mathematics.*

## AN ACCOUNT OF THE COMET OF DONATI. 1858.

BY G. P. BOND.

THE following account of the Great Comet, discovered by DONATI on the 2d of June, 1858, has been prepared for the Mathematical Monthly, with the intention of furnishing correct information on a subject of universal interest, in language freed as far as possible from technical expressions. The popular character of the article renders a few preliminary words of explanation necessary in reference to the more distinctive phenomena presented in the motions and physical aspect of comets generally.

The first characteristic of these singular bodies is that of their being mainly, perhaps in most instances, entirely composed of an ill-defined gaseous or nebulous substance, endowed with properties so extraordinary, that it can scarcely be classed with matter, in the ordinary acceptation of the term. Of its extreme attenuation and lightness, there can be no question. The planets, and among them our earth, must again and again have traversed unharmed the tails of comets. In October last, the debris of the magnificent train of the comet which has just disappeared from our western skies, swept over the region occupied by the earth a few weeks earlier. Instances of more immediate proximity are of too common occurrence to allow us to suppose that we are always to escape an actual collision; but it is inconceivable that any disastrous consequences could ensue to our earth or its inhabitants, any more than from contact with sunlight or with the ether of the planetary spaces.

A second characteristic is that of internal condensation. All comets present this in a greater or less degree. Most of them have a minute stellar point, called the nucleus, which occupies the position of maximum density. There are others in which this latter feature is wholly wanting. But the number, in which it cannot be detected with a powerful telescope, is much smaller than has com-

monly been supposed. This centre of condensation, or brightest point is, with rare exceptions, placed on the side which is nearest to the sun. It is always, however, very close to the centre of gravity, as is proved by the fact of its motion about the sun, in accordance with the law of gravitation.

The nucleus itself is a minute point compared with the immense volume of light-giving substance, of which it is the controlling centre. Whether it is solid or not, is still undecided. As far as the eye alone is to be trusted, there are comets as truly solid as the planets or stars themselves. In size and weight, however, the true nuclei, apart from their surrounding nebulosity, are probably quite small, measured by the standard of the larger planets. Still it is possible that there may have been instances in which the mass of these bodies has been comparable with that of the earth, and yet they may have completed their circuit around the sun, leaving no appreciable trace of their disturbing influence — the only sure test by which their mass could be detected. The evidence, from the fact that the smaller stars shine freely even through the most condensed portions of comets, adduced by astronomical writers in proof of their transparency, and, by inference, of their extreme tenuity and lightness, has a certain value when applied to the class of feeble telescopic comets; but is scarcely applicable to one like that of the present year, which overpowered all but the brighter stars in the neighborhood of the nucleus by its superior brilliancy.

The feature next in importance to the nucleus is the train, or tail, as it is usually called (although often preceding the nucleus in its motion), projected at an immense distance from it, and usually, although by no means invariably, in a direction opposite to that of the sun. The agency of the nucleus in the formation of the train, but still more in the subsequent control which it retains over it, is one of the most curious phenomena presented in nature. Often, several of these appendages are seen radiating at once from the

same nucleus. The greatest variety in curvature of outline, length, brilliancy, and other peculiarities, is presented by different comets, or by the same one in different parts of its course. The portions near the axis are usually darker than the edges, giving at times the appearance of a division with a stream of light on either side.

The larger bodies of this class exhibit a wonderful complication of phenomena in the region contiguous to the nucleus. Of these, the most prominent are the interposition between the nucleus and the sun of one or more well-defined and rounded screens, or caps of dense nebulosity, called envelopes, partially but not entirely surrounding the nucleus, and the emission of streams or jets of luminosity, bright sectors, &c., in a direction inclined or opposite to that of the tail. With great variety of detail in other respects, these have all a well-marked tendency to appear in the first instance on the side of the nucleus next the sun. The great comet of the present year undoubtedly takes a foremost rank in respect of the multiplied and most curious changes which it has exhibited, and especially in the complete illustration which it has afforded of the origin, construction, and final dissipation of a succession of envelopes. In these phenomena, the process of the formation of the tail, from the substance in immediate contact with the nucleus, is intimately concerned. The astronomer, night by night, sees the work of evolution going on with an amazing rapidity, and seemingly in defiance of the best established properties of matter, the laws of gravitation and of inertia. The results are evident to all, but the secret cause is a profound mystery admirably calculated to stimulate speculation and intelligent investigation.

The following cut, Fig. 1,* with a brief recapitulation, will serve

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* The engraving has been copied from a drawing of the comet of Donati, made by the writer, at the Observatory of Harvard College, October 2, 1858.

to illustrate the leading phenomena presented in a telescopic view of a large comet, *n* represents the star-like nucleus, *a* and *b* the outlines of two envelopes on the side of the nucleus towards the sun, *c* the diffused exterior nebulosity which is a never-failing attendant.

Fig. 1.

The dark axis, *d*, of the tail, is sometimes however absent, and again, at times, replaced by a bright ray. Ordinarily the convex and brightest side of the tail is presented to the region towards which the comet is moving.

As regards the motion of comets in space, it is a well-established fact, so far as our present means of observation extend, that their nuclei alone move in obedience to the attractive force of the sun and planets. This property, which has been recognized with consistency and uniformity, is not the least singular peculiarity of their constitution. Immense volumes of matter, apparently of the identical substance of the nucleus, go to compose the enveloping nebulosity and the tail, but from the moment of leaving the central body their motion is perfectly inexplicable without assuming them to be under the influence of laws of force which greatly modify that of gravitation.

The shape of the cometary orbits, described about the sun, is nearly that of a parabola, or of an elongated ellipse, with periods of revolution varying from a few years to many centuries. The point in the orbit which is nearest the sun is called the perihelion; the distance of this point from the sun, the perihelion distance, and the time of the comet's passing it, the perihelion passage.

Fig. 2.

Fig. 2 represents the form of a portion of the orbit of the great comet of the present year. The circles show the relative proportions of the orbits of Venus, the Earth, and Mars. The date, September 30, is the epoch of perihelion passage, the distance of the comet from the sun at that point being about fifty millions of miles.

For convenience of delineation the orbit is represented with its plane coincident with that of the earth's orbit. These planes have actually an inclination of sixty-three degrees to each other. The relative size and the form of the orbit, with the places of the comet in it, are given with sufficient exactness for the purpose of illustration. The arc described by the comet, during the period of its greatest brilliancy, is included between September 30 and October 10. It had, however, been detected by astronomers about three months earlier.

On the 2d of June, 1858, a faint nebulosity, slowly advancing towards the north, was descried by Donati at Florence, near the star $\lambda$ *Leonis*. This was the earliest observation of the great comet of 1858. Its place at the time is shown in Fig. 2, its distance from the sun being then about two hundred millions of miles, while from the earth it was yet more remote. Being, at first, inclined to question whether it might not be identical with another comet just before seen in the same quarter of the heavens (the third comet of 1858), he communicated the intelligence of the discovery with a suitable reserve, as "perhaps new;" and in a second despatch he

said, "It is possible that this comet is the same as that discovered in America on the 2d of May." This conjecture, fortunately for DONATI, did not prove true; although the apprehension of the Italian astronomer, from the rival zeal of his transatlantic brethren, was not without reasonable foundation, for no sooner had the moon withdrawn from the evening sky so as to allow the comet to be seen, than it was detected almost simultaneously at three different points in America, each observer being at the time unaware of its previous discovery in Italy. It was seen by Mr. H. P. TUTTLE on the evening of the 28th of June, and an accurate determination of its place was made on the same night at the Observatory of Harvard College. On the 29th, it was detected by H. M. PARKHURST, Esq., of Perth Amboy, N. J., and on the 1st of July, by Miss MITCHELL, of Nantucket.

Three geocentric positions obtained on the 7th, 11th, and 13th of June, furnished DONATI with the means of computing approximate elements of the comet's motion, from which its interesting character was quickly recognized. Considerable difficulty was experienced, in fixing the precise time of perihelion passage, a most necessary condition in predicting its path as seen from the earth. While in other respects the results deduced by various computers were sufficiently accordant, they showed wide discrepancies in designating the place of the comet in the orbit. By the middle of August, however, its future course, and great increase of brightness in September and the early part of October, had been ascertained with entire certainty.

Up to this time it had remained a faint object, not even discernible by the unassisted eye. It was distinguished from ordinary telescopic comets only by the extreme slowness of its motion, in singular contrast with its subsequent career, and by the vivid light of the nucleus; the latter peculiarity was of itself prophetic of a splendid destiny.

Traces of a tail were noticed on the 20th of August, and on the 29th it was seen with the naked eye as a hazy star. For a few weeks it occupied a position in the heavens where it rose before the sun and set after it, becoming thus a conspicuous object both in the morning and evening sky. This circumstance gave rise to the erroneous notion that two different comets had appeared. The statement, which was widely circulated, that this was the return of the comet of 1264 and of 1556, supposed by some to be identical, is equally incorrect. If it has ever before been seen by man, it must have been far back in history, since the most recent computations assign a time of revolution of about twenty-five hundred years.

On the 6th of September was first noticed the curvature of the tail, which subsequently, at the time of its greatest expansion, became one of its most impressive features. It is remarkable that this peculiarity should have been strongly enough exhibited to be distinguished at the above date, when the earth was close to the plane of the comet's orbit. The observation cannot in fact be reconciled with the commonly received opinion that the curvature of the tail lies in the plane of motion about the sun.

The extraordinary changes exhibited in the neighborhood of the nucleus during the disruption of the envelopes, and by the train while the comet was in the part of its orbit nearest the sun, will be illustrated in the conclusion of this article by a series of woodcuts and of mezzotint plates. This method of description is the only one by which an adequate conception of the condition of the comet, during this critical period in its history, can be conveyed to the reader. Fortunately, a long succession of clear skies at Cambridge afforded uncommonly favorable opportunities for observation.

# Editorial Items.

UNDER this head we shall include all such miscellaneous matter as we wish to communicate to the readers of the Monthly, and for which we do not find a more appropriate place. ..... PRIZE PROBLEMS FOR STUDENTS. — Since the issue of the first number of the Monthly, we have received several letters, and a complete set of solutions of the prize problems, from those who are not connected with any institution of learning, and yet wish to compete for the prizes. When we first decided to offer these prizes, we were desirous not to exclude any one justly entitled to compete for them, although he might not be so fortunate as to be connected with an institution of learning; but at that time it did not occur to us how we might extend them to all students, and still secure entire fairness among all the competitors. In the October number, we stated that the slips, containing the names of the competitors, and the institutions with which they are connected, must also be signed by their instructors, as evidence that the parties are fairly entitled to compete for the prizes. Now, if any student, not connected with an institution of learning, whether public or private, who wishes to compete for any of these prizes, will send the statement of a responsible person giving us the evidence that he is fairly entitled to be considered a competitor, we will withdraw the limitation...... It gives us pleasure to add the following names to our list of coöperators and contributors. *Major R. J. ADCOCK, Military Inst., Ky.; *Prof. WM. W. CLARKE, Genesee Wesleyan Seminary, Lima, N. Y.; J. N. LEWIS, Esq., Uniontown, Penn.; Rev. J. P. PERRY, Yarmouthport, Mass.; C. M. RUNK, Esq., Allentown, Penn.; EDWARD W. SERRELL, C. E., New York; P. H. THOMSON, C. E., Columbia, Tenn...... In the October number of the Monthly, on page 2, the right hand member of the first equation of Prize Problem I. should be — 2 instead of + 2. This will make the equation of the second degree as was intended. On page 19, lines —1 and —2, for $\varrho_z$ read $\varrho_x$; page 19, lines —3 and —4, for $\varrho_z$ read $\varrho_y$; page 21, line 10, for $1 + (D_x y)^2$ read $(1 + (D_x y)^2)^{\frac{1}{2}}$. These errata are corrected in the second edition...... Prof. PEIRCE's propositions on the distribution of points on a line have excited unusual interest. We have already received demonstrations from R. J. ADCOCK, W. P. G. BARTLETT, PLINY EARLE CHASE, E. B. ELLIOTT, SIMON NEWCOMB, and JAMES EDWARD OLIVER. They will receive early attention...... Contributions intended for any particular month should be received at least one month in advance...... A resolution, in favor of the Mathematical Monthly, moved by JAMES CRUIKSHANK, Esq., Editor of the New York Teacher, was passed by the New York State Teachers' Association at its late session, held at Lockport, N. Y. The certified copy ordered to be sent failed to reach us...... We wish to call the attention of Engineers to the Mathematical Monthly, and we trust that in the numbers already issued the evidence is not entirely wanting of its educational and practical value to their profession. See, for instance, the articles on Ovals and Three-Centre Arches, and on the Prismoidal Formula and its Extension...... We wish to call the attention of authors and publishers (especially of mathematical and scientific books) to the Mathematical Monthly as an advertising medium...... Books received. Elements of Mechanics. By J. B. CHERRIMAN, M. A., late Fellow of St. John's College, Cambridge, and Professor of Natural Philosophy in University College, Toronto. Maclear & Co., King Street East. 1858. Nouvelles Annales de Mathématiques for September, 1858. Paris: Mallet-Bachelier. Mensuration and Practical Geometry. By CHARLES H. HASWELL, Civil and Marine Engineer. New York: Harper & Brothers, Publishers, Franklin Square. 1858.

THE

# MATHEMATICAL MONTHLY.

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## PRIZE PROBLEMS FOR STUDENTS.

### I.

In every triangle, the perpendiculars dropped from the angles to the opposite sides meet in the same point; the perpendiculars erected to the middle of its sides meet in the same point; the lines drawn from the angles to the middle of the opposite sides meet in the same point. Prove that these three points are in the same straight line, and that their distances apart are in a constant ratio.

### II.

Prove that $\tan 9^0 = \sqrt{5} + 1 - \sqrt{5 + 2\sqrt{5}}$.

### III.

Find the coefficient of $x^n$ in the expansion of $\dfrac{ax + b}{(x-1)(x+2)(x-3)}$ into a series of ascending powers of $x$.

### IV.

The equations $y = ax + a + 1$, $y = (a + 1)x + b$, and $y = bx + 5$, are equations of straight lines in which $a$ and $b$ are positive integers, and the three lines all meet in one point. What values can $a$ and $b$ have, and what are the corresponding coördinates of the common point of the lines?

V.

When the angle between the coördinate axes is $\frac{\pi}{3}$, prove that $x + y - a = \sqrt{x y}$ is the equation of a circle of which the radius is $\frac{1}{3} a \sqrt{3}$.

The solutions of these problems must be received by February 1st, 1859.

## ARABIC NOTATION IN MENTAL ARITHMETIC.

BY W. W. NEWMAN.

In all written or practical arithmetics the Arabic notation is introduced and used with the first lessons, because the numbers are so large that the operations would become exceedingly difficult, if pupils were confined to the English printed or written words which name the numbers.

But in mental or intellectual arithmetics there is a diversity of practice, and, of course, a difference of opinion among our most popular authors. Colburn, the celebrated pioneer in this class of works, first uses the figures from 1 to 10 on the fiftieth page, after going through with the simple rules and an introduction to fractions. He also explains the Arabic notation of the numbers from 10 to 100 on the sixty-ninth and seventieth pages; and, with remarkable coincidence, Adams, Perkins, and Thomson do the same at precisely the same place.

On the contrary, Davies, in his New Primary, Greenleaf, Robinson, Stoddard, Emerson, &c., introduce the pupil immediately to the language and practice of the Arabic notation.

Colburn says figures "are not used in the first part of the book, because the pupil would not understand them so well as he will the words," and this is probably the idea of other authors.

But it should be remembered that teachers of the simplest reading lessons find it necessary to teach their pupils the Arabic notation that numbers the pages of their books, and, therefore, the arithmetical language of so small numbers is generally learned before they are introduced to even the simplest primary arithmetic. Again, all the numbers below ten, and all units in larger numbers are presented to the eye by the Arabic notation with a single character; but the shortest of the words has three, and the longest five letters. The tens figures, using only one *figure*, require from four to seven *letters* each. If, therefore, a child cannot understand the figures 8, 16, 98, as well as the English words *eight*, *sixteen*, *ninety-eight*, when they are read alike, and are only two forms of expressing the same things, it must be that a brief, simple mode of spelling only befogs the juvenile intellect, and that silent, unnecessary letters are aids that cannot be dispensed with in primary instruction. Figures are more easily read than words; they are more rapidly written upon the blackboard or slate; they give a condensed and expressive view of operations; they are great improvements upon all previous modes of expressing even small numbers; and the pupil may, without any hinderance, delay, or injury, be introduced to his arithmetical alphabet of ten Arabic characters or letters as his first lesson in the science of numbers. If there are any good and sufficient reasons why these characters should be deferred to the advanced portions of mental arithmetic, we have yet to learn what they are.

## DIVISION OF FRACTIONS.

BY A. SCHUYLER.

THIS subject is usually presented in three cases: — I. To divide a fraction by a whole number. II. To divide a whole number by a fraction. III. To divide a fraction by a fraction.

Though the subject may properly be treated under the above heads, yet it should be understood, that the last case is general, including the other two; for, any whole number can be expressed in the form of a fraction having any denominator. Thus, in general, $n = \frac{n}{1} = \frac{n\,d}{d}$. We shall therefore treat of the general case, III., by reducing the operation to a division of whole numbers.

PRINCIPLES. 1. *If any simple fraction be multiplied by any multiple of its denominator, the result is a whole number.*

Thus, in general, let $\frac{n}{d}$ be any simple fraction, and $m$ any whole number, then $m\,d$ will represent any multiple of the denominator. Then, $\frac{n}{d} \times m\,d = m\,n$, a whole number, since the product of two whole numbers is a whole number.

2. *If both dividend and divisor be multiplied by the same number, the quotient will remain unchanged.* Thus, in general, let $D$ denote the dividend, $d$ the divisor, then $D \div d = D \times m \div d \times m$.

Hence, *If both dividend and divisor be multiplied by any common multiple of their denominators, they will become whole numbers, the quotient will remain unchanged, and will be expressed by a fraction, of which the new dividend is the numerator, and the new divisor the denominator.*

To make the operation as concise as possible, multiply both dividend and divisor by the least common multiple of their denominators.

Let it be required to divide $\frac{3}{4}$ by $\frac{5}{6}$.

OPERATION.

$\frac{3}{4} \div \frac{5}{6} = 9 \div 10 = \frac{9}{10}$

By multiplying both dividend and divisor by 12, the least common multiple of their denominators, 4 and 6, we have $9 \div 10 = \frac{9}{10}$.

Complex and compound fractions should be reduced to simple fractions before division, but mixed numbers need not be thus reduced. Thus, $5\frac{1}{2} \div 2\frac{1}{4} = 22 \div 9 = 2\frac{4}{9}$.

In multiplying both dividend and divisor by the least common multiple of their denominators, cancel as much as possible; and even this should be done mentally, and we should, at once, say

$\frac{3}{4} \times 12 = 9$. In general, $\frac{n}{d} \div \frac{n'}{d'} = n\,d' \div n'\,d = \frac{n\,d'}{n'\,d} = \frac{n}{d} \times \frac{d'}{n'}$, which demonstrates the rule, *Invert the divisor, and proceed as in multiplication.*

It is a serious objection to this rule, that its tendency is to lead the scholar to rely on a mere mechanical process, in which case he soon loses sight of the principle. But the above method keeps the principles constantly before the mind. Let these principles first be thoroughly understood, and the above method will be found very concise and beautiful.

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## NOTES, BY PROF. ELIAS LOOMIS.

I. My own experience has led me to a very different conclusion respecting decimal fractions from that stated by Mr. Hill, page 3. Instead of "keeping sedulously out of sight the analogy between decimal and vulgar fractions," I would insist that the pupil should constantly remember, not the *analogy* between decimal and vulgar fractions, but the essential *identity* of vulgar fractions with decimals and duodecimals; and for this purpose I would recommend that the pupil be always required to write decimal fractions and duodecimals with a denominator, until he becomes perfectly familiar with the idea that the omission of the denominator is simply a matter of convenience, and only to be allowed in cases where its omission leaves no uncertainty as to what the denominator should be.

II. Mr. Safford, on page 10, has not proved that one of Napier's rules is superfluous, but simply that one rule may be deduced from the other. Napier's rules never professed to inform us of principles not known before, but were designed to embody in a form easily remembered and easily applied, the theorems for the solution of every case in right angled spherical triangles. In this point of view neither of Napier's rules is superfluous, and the two rules together

probably form the most fortunate application of the principles of artificial memory, which the history of the mathematics can furnish.*

## ON THE STUDY OF GEOMETRY.

BY PROF. M. C. STEVENS.

THE sentiments expressed by SAMUEL P. BATES, in the first number of the Monthly on the study of Geometry, I consider worthy of the attention of every teacher. Let teachers try the experiment of giving to their classes in geometry, after they have gone over the text-book, some simple problem or theorem not found in the book, and I venture the prediction of a failure in nine cases out of ten,— provided they have not been accustomed to rely on their own powers. If this be so, what does it prove? Most assuredly, that there has been some error in the mode of teaching.

It is my opinion that *pure geometry* receives too little attention in our schools generally; and that the analytical method of arriving at geometrical truths is taking too prominent a place. I think I have a just sense of the importance of analysis and of its power; but I want *pure geometry* to occupy the place it deserves as a means of mental discipline, as well as for its beauty and brevity in the solution of problems. But it is urged that the analytical method is more powerful, because we are led directly to our result; whereas it frequently takes much time to discover the hidden relations that exist in a problem, which must be known in order to apply the other method. All this I am ready to grant, but I urge that greater facility may be acquired in discovering these hidden relations than

* We suppose that Mr. SAFFORD only intended to show that Napier's rules are not analytically independent; and not that Rule I. should be discarded in practice. If so, then there is no disagreement between Mr. SAFFORD and Prof. LOOMIS. — ED.

is generally supposed. Hence I say, let geometry be taxed to the extent of its power, and not thrown aside too soon.

To this end I think every instructor in mathematics should propose a great variety of geometrical problems to his class for construction. Of course the problems should be selected in reference to the capacity of the class, at first selecting those that are simple, and letting them increase in difficulty by degrees, as the class is able to master them. I have intentionally used the term analysis and geometry in a limited sense. And to show what I really mean by the term geometrical construction, I here introduce the solution to the following problem.

Through a given point without a given angle to draw a straight line, so as to cut off a given area.

CONSTRUCTION.


Let $MAN$ be the given angle, and $P$ the given point. Draw $PE$ parallel to $AM$, and make the parallelogram $ADEF$ contain the given area. Draw $DG$ perpendicular to $AM$, and equal to $PF$. Make $GB = PE$, and join $PB$, and $ABC$ is the triangle required. Demonstration. The triangles $PHE$, $PCF$, and $BDH$ are evidently similar; and by construction $PE^2 - PF^2 = DB^2$. Hence $PHE - PCF$, or $HECF = BDH$. Therefore $ABC = ADEF$. *Q. E. D.*

The reader may compare this with the solution of the same problem as given in Church's Analytical Geometry, and draw his own conclusions. The inference is not to be drawn, however, that the solution there given is not a good one of its kind, and well calculated to illustrate what is intended.

## ON THE GENERAL PROPERTIES OF EQUATIONS.

BY L. W. MEECH.

At a certain stage of progress, the student of Algebra, writing down an equation of the third or fourth degree with numeric coefficients taken entirely at pleasure, would suppose there must be some value of $x$ which satisfies the equality. But he would have no adequate idea of the extent to which such coefficients in relation to the roots of the equation are subject to uniform laws, and completely pervaded by them in every part. It was by bringing all the terms to one side so as to make them equal to zero, that Harriot, the companion of Sir Walter Raleigh, in Virginia, first generalized what Cardan and Vieta had well commenced. He showed that the unknown quantity in an equation always has as many values or roots as the index of its power in the first term denotes; he likewise unfolded the composition of the coefficients, and proved them to be represented by the roots taken in regular combinations. The perception of these and other relations has tended to free the science from the amusing enigmas and circumlocutions of the older writers; and widening its sphere has given perspicuity of thought and style to its investigations.

For a practical exercise in these principles, let it be proposed to demonstrate the following new indication of imaginary roots; the given equation being expressed under the general form, with real coefficients,

$$Ax^n + Bx^{n-1} + Cx^{n-2} \dots + C'x^2 + B'x + A' = 0.$$

*Whenever* $(n-1)\,B^2 - 2\,n\,A\,C$ *is negative, or when* $(n-1)\,B'^2 - 2\,n\,A'\,C'$ *is negative, either or both, there are two or more imaginary roots.*

The proof of this relation is here subjoined in its leading features; a constant reference being presupposed to the latter chapters of any of the works on Algebra which treat of the General Proper-

ties of Equations. Dividing the preceding equation through by $A$, it is plain, if the roots are all real, the sum of their squares must be positive; since the square root of a negative quantity is imaginary. This sum of the squares is shown in Algebra to be represented by $\frac{B^2}{A^2} - \frac{2C}{A}$; and if this sum be negative, the circumstance must arise not from real, but from imaginary roots, or from a preponderance of the latter. Now to draw this restriction more closely, let us take away the second term of the given equation, by assuming $x = y - \frac{B}{nA}$; in this case the sum of the squares of the values of $y$ will become $\frac{(n-1)B^2 - 2nAC}{nA}$, by proceeding as before. If this also be negative, imaginary roots are the cause; and the denominator may be removed, which does not alter the sign. Lastly, substituting $\frac{1}{z}$ for $x$, and clearing the given equation of fractions, the preceding reasoning on the first three terms is likewise applicable to the last three terms, and the proposition is demonstrated.

When the coefficients are real, "imaginary roots always enter in pairs," and if $n$ is made equal to 2 in the preceding criteria, we have essentially the same test of imaginary roots, which is presented by the common solution of the quadratic equation. It should further be remarked, that there is no simple and universal criterion of imaginary roots; the preceding test, as well as Descartes' rule of signs, admits of occasional application.

## NOTE ON THE LEVER.

BY R. G. HATFIELD.

THAT the weight and the power, when in equilibrio, are in proportion, inversely, as their respective distances from the fulcrum, may be demonstrated as follows:—

A rod of uniform section, and of some homogeneous material

resting at its centre on a point of support, as at $C$, Fig. 1, may be supposed to be divided at some point $D$, into two unequal lengths, and the weight of these two unequal portions may be supposed concentrated at their respective centres of gravity $E$ and $F$, without destroying the equilibrium. This is represented in Fig. 2. The line $EF$ being supposed rigid and without weight: $W$, being equal to the weight of the part of the rod $AD$, and $P$, to the part $DB$.

Fig. 1.

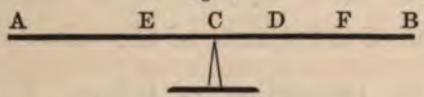
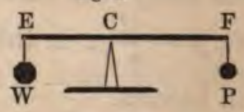



Fig. 2.




Thus we have the power $P$, acting upon the weight $W$, through the lever $EF$, and the power and weight are in a state of equilibrium. Now it is to be shown that $W : P :: CF : EC$; or, since $W$ equals the weight of $AD$ and $P$, the weight of $DB$, and since, as the rod is uniform throughout its length, the lengths of the parts are in proportion as their weights, and may be taken to represent their weights respectively, therefore it will be sufficient to show that $AD : DB :: CF : EC$. Since $ED$ is the half of $AD$, and $DF$ the half of $DB$, therefore $EF$ is the half of $AB$; hence $EF$ is equal to $AC$ or to $CB$.

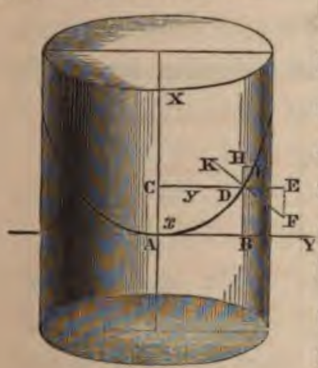
First, $EF = AC$: and these are each equal the sum of their parts, or, $EC + CF = AE + EC$. Cancelling the identical terms, there remains $CF = AE$; and since $AE = \frac{AD}{2}$, therefore, $CF = \frac{AD}{2}$.

Second, $EF = CB$ as above shown; and these are each equal the sum of their parts, or, $EC + CF = CF + FB$. Cancelling the identical terms, there remains $EC = FB$; and since $FB = \frac{DB}{2}$, therefore, $EC = \frac{DB}{2}$. Now since $EC$ equals the *half* of $DB$, and $CF$ equals the *half* of $AD$ (as above), therefore is $EC : DB :: CF : AD$, and by transposing the extremes, $AD : DB :: CF : EC$.

## A FLUID PARABOLIC MIRROR.

BY PROF. GEORGE R. PERKINS.

*If an open vertical cylinder, containing a fluid, is made to revolve with a uniform motion about its axis, the upper surface of the fluid will assume a perfect parabolic form.*




If we consider a particle of the fluid at $D$, it must, in reference to the surrounding particles, remain in equilibrium. Hence, the resultant of the forces acting upon this particle must be in the direction of the normal to the surface assumed by the fluid, that is, it must be in the direction of $DF$. Consequently the force of gravity must be to the centrifugal force as $EF$ is to $ED$. If we call the coördinates of this point $x$ and $y$, we shall have in the differential triangle $DHG$, $HD = dx$; $HG = dy$; but the triangle $DHG$ is similar to the triangle $DEF$, hence, $dy : dx ::$ force of gravity : centrifugal force. Denoting, as usual, the force of gravity by $g$, and by $T$ the time in seconds of one revolution of the cylinder, we shall have $\frac{2\pi y}{T}$ for the velocity, of the point at $D$. But the centrifugal force is expressed by the square of the velocity divided by the radius, that is, we have $\left(\frac{2\pi y}{T}\right)^2 \div y = \frac{4\pi^2 y}{T^2}$, for the centrifugal force. Hence, the above proportion becomes

$$dy : dx :: g : \frac{4\pi^2 y}{T^2} \text{ or,}$$

$$\frac{4\pi^2 y\,dy}{T^2} = g\,dx, \text{ that is, } 2y\,dy = \frac{g\,T^2\,dx}{2\pi^2}.$$

Integrating, we have

$$y^2 = \frac{g\,T^2 x}{2\pi^2}.$$

This is the equation of a Parabola having $\frac{g\,T^2}{8\pi^2}$ for its focal dis-

tance. If we denote by $N$ the number of revolutions in each minute, we shall have $N = \frac{60}{T}$.

Calling $g = 32.1908$ feet; $\pi = 3.14159$, we shall have the focal distance in feet expressed by $T^2 \times 0.4073 \ldots = \frac{1466.33 \ldots}{N^2}$. From this expression we see that the focal distance varies inversely as the square of the angular velocity.

We deduce from the above expression the following numerical results: —

| No. | Focal distance in feet. | No. | Focal dist. in feet. |
|---|---|---|---|
| 1 | 1466.33 | 20 | 3.66 |
| 2 | 366.58 | 30 | 1.63 |
| 3 | 162.92 | 40 | 0.92 |
| 4 | 91.64 | 50 | 0.59 |
| 5 | 58.65 | 60 | 0.41 |
| 6 | 40.73 | 70 | 0.30 |
| 7 | 29.92 | 80 | 0.23 |
| 8 | 22.91 | 90 | 0.18 |
| 9 | 18.10 | 100 | 0.15 |
| 10 | 14.66 | 120 | 0.10 |

From these results we see that when the fluid revolves sixty times in one minute, that is, once in each second of time, the focal distance will be 0.41 feet = 4.92 inches. If it revolve once in six seconds, the focal distance will be $14\frac{2}{3}$ feet nearly. If it revolve once in ten seconds, the focal distance will be $40\frac{3}{4}$ feet, nearly.

*Is it possible to make use of this kind of mirror for Astronomical purposes?*

In theory, this fluid mirror will be perfect in its parabolic form, whatever the velocity of revolution may be, provided it is uniform. Each and every ray of light falling upon it parallel to its axis will be reflected to the same focal point. Whatever change the fluid may undergo by reason of a change in the atmospherical temperature cannot alter the parabolic form of the surface, since this form does not depend at all upon the specific gravity of the fluid, but

only upon the velocity of rotation, consequently this mirror will remain perfect at all temperatures, which must give it great advantages over the ordinary solid mirrors.

And since the focal distance does not depend upon the size of the mirror, it follows, theoretically, that there is no limit to the size of mirrors constructed in this way, consequently any amount of light may be used which will allow of high magnifying powers in the eye-glasses, and in this consists the great telescopic power which we shall in this way be able to obtain.

As yet, we have given no method by which objects out of the zenith can be viewed.

This must be accomplished by first receiving the rays of light upon a plane mirror, so adjusted as to throw them after reflection vertically upon the parabolic mirror. This plane mirror will of necessity absorb more or less of the direct rays, still as the size of the mirror is unlimited, I suppose no serious objection would result from this. I apprehend the most difficult thing to be overcome in this case is to construct a plane mirror of such vast size which shall be sufficiently accurate to answer our purpose. For this plane mirror must be solid, and subject to all the defects incident to mirrors so formed.

But suppose this fluid parabolic mirror can only be employed in a zenith telescope, does it not promise advantages sufficiently great to warrant the expense of all necessary experiments to test its practicability?

By using mercury, which possesses the speculum surface but little short of perfection, and which also is of such specific gravity as to become very stable after having once assumed its position of equilibrium, it would seem that in the present state of mechanical perfection we may reasonably hope to succeed in constructing in this way reflecting telescopes of almost unlimited power.

Should this plan succeed, I need not point out its great importance in a scientific point of view. The astronomer will then have the ability of deciding many questions in regard to the theory of nebulous bodies, to say nothing about the distinctness with which it must of necessity expose to view the surface of the moon. The question so often asked, as to whether the moon is inhabited, may then be at once answered without a possibility of error.

Some of the most perfect mirrors which it is possible for us to conceive of, are formed by the physical laws of nature, in the atmosphere, in the case of *mirage*. This fact is well calculated to encourage the hope of success in the case of mercurial parabolic mirrors.

## DEMONSTRATION OF PROF. PEIRCE'S PROPOSITIONS ON THE DISTRIBUTION OF POINTS ON A LINE.

BY MAJOR R. J. ADCOCK.

PROPOSITION (1). Let the number of points upon a line whose length is $a$ be represented by $a$, and the number upon any part $x$ measured off from one end by $x$. If the line $x$ move along the line $a$ until its forward extremity reach the other end of $a$ the two points at the extremities of $x$ will evidently have taken $(a - x)$ positions upon $a$. Hence the whole number of possible positions of two points upon $a$, which are never at a greater distance than $x$, equals the sum of all the remainders $(a - x)$ obtained by varying $x$ from 0 to $x$. This sum $= \int_0^x (a - x)\, dx = a x - \frac{1}{2} x^2$. The whole number of possible relative positions of two points upon $a$ is $\int_0^a (a - x)\, dx = \frac{1}{2} a^2$. When $a x - \frac{1}{2} x^2 = \frac{1}{2}$ of $\frac{1}{2} a^2$, $x = a (1 \pm \frac{1}{2} \sqrt{2}) = 0.29289 \times a$.

Prop. (2). When $x = \frac{1}{2} a$, $a x - \frac{1}{2} x^2 = \frac{3}{8} a^2 = \frac{3}{4} \times \frac{1}{2} a^2$.

Prop. (3). When $x = \frac{1}{3} a$, $a x - \frac{1}{2} x^2 = \frac{5}{18} a^2 = \frac{5}{9} \times \frac{1}{2} a^2$.

Prop. (4.) When $x = \frac{2}{3} a$, $a x - \frac{1}{2} x^2 = \frac{8}{18} a^2 = \frac{8}{9} \times \frac{1}{2} a^2$.

Prop. (5). Since, when the line returns into itself, the greatest distance of two points can be only $a$, two points at a distance $x$, can only take $a$ positions; the sum of which from 0 to $x = \int_0^x a\, d x = a x$.

Prop. (6). The number of positions of two points upon $x$ is $\frac{1}{2} x^2$.

The number of these, with another point on $d x$, is $\frac{1}{2} x^2 d x$ positions of three points. Hence the whole number of positions of three points upon $x$ is $\int_0^x \frac{1}{2} x^2 d x = \frac{1}{6} x^3$. When the line returns into itself, the longest possible distance of two points may still be regarded as $a$, and the whole number of positions of three points upon it, is $\frac{1}{6} a^3$.

Let $x$ be the least distance of the two nearest points, then $2 x$ is the least distance that can be occupied by three points. Beginning with this position, let the two forward points, remaining at the same distance, move around the curve until the advance point is at the distance $x$ from the fixed point. The three points will thus have taken $(a - 3 x)$ positions. Hence $\int_0^x (a - 3 x)\, d x$ is the number of positions of three points when one is fixed, and the other two never at a greater distance than $x$, nor than either of their distances from the fixed point. But the fixed point can be moved a distance $a$ from its first position. Hence the whole number of positions of three points under the conditions is $a \int (a - 3 x)\, d x = a^2 x - \frac{3}{2} a x^2 + C$. Taking the integral between the limits 0 and $x$, we have $a^2 x - \frac{3}{2} a x^2$ for the number of positions when the two nearest points are at a less distance than $x$. When $a^2 x - \frac{3}{2} a x^2 = \frac{15}{16} \times \frac{1}{6} a^3$, $x = (\frac{1}{3} \pm \frac{1}{12})\, a = \frac{5}{12} a$, or $\frac{1}{4} a$.

PROP. (7). The constant of integration in $a^2 x - \frac{3}{2} a x^2 + C$, is $\frac{1}{6} a^3$, for when the least distance of the two nearest points is 0, the whole number of positions is $\frac{1}{6} a^3$. When $a^2 x - \frac{3}{2} a x^2 + \frac{1}{6} a^3 = \frac{1}{2} \times \frac{1}{6} a^3$, $x = \left(\frac{1 \pm \frac{1}{2}\sqrt{2}}{3}\right) a \doteq 0.09763\, a$.

PROP. (8). When $a^2 x - \frac{3}{2} a x^2 = \frac{1}{6} a^3$, $x = (\frac{1}{3} \pm 0)\, a$.

When $a^2 x - \frac{3}{2} a x^2 = \frac{3}{4} \times \frac{1}{6} a^3$, $x = (\frac{1}{3} \pm \frac{1}{6})\, a = \frac{1}{2} a$ or $\frac{1}{6} a$.

When $a^2 x - \frac{3}{2} a x^2 = \frac{63}{64} \times \frac{1}{6} a^3$, $x = (\frac{1}{3} \pm \frac{1}{24})\, a = \frac{3}{8} a$ or $\frac{7}{24} a$.

The three points are evidently always included in an arc equal to $a$, minus the greatest distance of two points in any case.

Hence the arc which includes the positions $\frac{1}{6} a^3$ is $\frac{2}{3} a$.

Hence the arc which includes the positions $\frac{3}{4} \times \frac{1}{6} a^3$ is $\frac{1}{2} a$.

Hence the arc which includes the positions $\frac{63}{64} \times \frac{1}{6} a^3$ is $\frac{5}{8} a$.

Since by Prop. (5), the number of positions of two points included in an arc $x$, less than $a$, is $a x$, the number of positions of three points included in $x$ is $\int a x\, dx = \frac{1}{2} a x^2$.

These propositions might be demonstrated by the Theory of Permutations and Combinations.

## THEORY OF THE DISTRIBUTION OF POINTS ON A LINE.

BY W. P. G. BARTLETT.

LET $l$ be the length of a line returning into itself, and $dl$ the distance between two of its consecutive points. Let it be divided into $n$ intervals by $n$ points. Then obviously the coefficient of $x^l$ in the development of

(1) $$[x^0 + x^{dl} + x^{2dl} + x^{3dl} \dots\dots + x^{\frac{l}{dl} dl}]^n,$$

will express $n!$* times the number of different sets of $n$ intervals

* Using the notation $n!$ to express the product $1.2.3.4\dots\dots n$

into which the line may be divided. Putting $x^{dl} = z$ and $\frac{l}{dl} = p$, (1) becomes

$$(2)\ [1+z+z^2+z^3+\ldots..+z^p]^n = \left(\frac{1-z^{p+1}}{1-z}\right)^n = (1-z^p)^n(1-z)^{-n}$$
$$= (1-A_1 z^p + A_2 z^{2p} - A_3 z^{3p} + \ldots.. \pm A_n z^{np})$$
$$(1-B_1 z + B_2 z^2 - B_3 z^3 + \ldots..),$$

in which $A$ and $B$ are the binomial coefficients for positive and negative values of $n$ respectively.* The coefficient of $x^l = z^p$ in (2) is $\pm B_p - A_1 = \pm B_p = \frac{(n+p-1)!}{p!\,(n-1)!} = \frac{p^{n-1}}{(n-1)!}$.

$$(3)\qquad \therefore \frac{p^{n-1}}{n!\,(n-1)!} = \frac{l^{n-1}}{n!\,(n-1)!\,d\,l^{n-1}}$$

is the number of different sets of $n$ intervals into which the line $l$ can be divided.

To find in how many of these sets the least interval is $x$, we may suppose the excess of $l$ over $n\,x$ to be distributed in all possible ways (no reference being had to the order of the parts), among the remaining $n - 1$ intervals, giving therefore

$$(4)\qquad \frac{(l-n\,x)^{n-2}}{(n-1)!\,(n-2)!\,d\,l^{n-2}}$$

different sets of $n$ intervals, in each of which the least interval is $x$. But as the different orders in which each set of intervals may be distributed over the line depend only on their number $n$, we may divide (4) by (3) and obtain

$$(5)\qquad \frac{n\,(n-1)\,(l-n\,x)^{n-2}\,d\,l}{l^{n-1}},$$

as the *probability* of the least one of the intervals being $x$. Putting $d\,x$ for $d\,l$, we may integrate (5) with reference to $x$ between any limits greater than 0 and less than $\frac{l}{n}$, and thus obtain as the *probability* of the least interval being included between $a$ and $b$,

$$(6)\qquad \int_a^b \frac{n\,(n-1)\,(l-n\,x)^{n-2}}{l^{n-1}}\,d\,x = \frac{(l-n\,a)^{n-1} - (l-n\,b)^{n-1}}{l^{n-1}}.$$

* This method I derived from the Introduction to BREMIKER's Six Place Logarithms.

When $a = 0$, (6) becomes $\frac{l^{n-1} - (l - nb)^{n-1}}{l^{n-1}}$, which may also be obtained by distributing the excess $l - nb$ in all possible ways (without regard to the order of the parts), among the $n$ intervals, and taking its complement to unity.

To find in how many of the sets (3) all the points are just included in a length $y$, we may suppose $y$ to be divided in all possible ways (without regard to the order of the parts), into $n - 1$ parts, *provided* that no part shall be greater than $l - y$. Then (2) becomes, putting $\frac{y}{dl} = q$,

$$(7) \quad [1 + z + z^2 + \ldots\ldots + z^{p-q}]^{n-1} = (1 - A'_1 z^{p-q} + A'_2 z^{2(p-q)} - A'_3 z^{3(p-q)} + \ldots\ldots \pm A'_{n-1} z^{(n-1)(p-q)}) (1 - B'_1 z + B'_2 z^2 - B'_3 z^3 \ldots\ldots),$$

in which $A'$ and $B'$ are the binomial coefficients for the $(n-1)$st power, and the coefficient of $z^q$, divided by $(n-1)!$, is the number of ways in which $y$ is required to be divided. The general form of this coefficient, retaining only the infinite terms, is

$$\pm B'_q \pm A'_1 B'_{2q-p} \pm A'_2 B'_{3q-2p} \pm A'_3 B'_{4q-3p} \pm, \text{ \&c.}$$

$$= \frac{q^{n-2}}{(n-2)!} - (n-1) \frac{(2q-p)^{n-2}}{(n-2)!} + \frac{(n-1)(n-2)}{2!} \cdot \frac{(3q-2p)^{n-2}}{(n-2)!} - \text{ \&c.},$$

in which there can be no more than $n$ terms, and only those are to be retained in which the subscript number to $B'$ is positive. Whence dividing by $(n-1)!$ and by (3) and substituting the values of $p$ and $q$, we get, since $dy = dl$,

$$(8) \qquad \frac{n(n-1) y^{n-2}}{l^{n-1}} dy - \frac{n(n-1)^2 (2y-l)^{n-2}}{l^{n-1}} dy + \frac{n(n-1)^2 (n-2)(3y-2l)^{n-2}}{2!\, l^{n-1}} dy - \text{ \&c.},$$

as the *probability* (omitting the terms which become negative), of all the points being included in a length $y$. If we integrate (8), taking care to include no negative term between the limits, and remembering that $y$ cannot be greater than $\frac{(n-1)l}{n}$, we have

$$(9)\qquad \int_a^b \frac{n(n-1)\,y^{n-2}}{l^{n-1}}\,dy - \int_{a'}^b \frac{n(n-1)^2\,(2y-l)^{n-2}}{l^{n-1}}\,dy + \int_{a''}^b \frac{n(n-1)^2\,(n-2)\,(3y-2l)^{n-2}}{2\,!\,l^{n-1}}\,dy - \text{\&c.},$$

in which $b < \frac{(n-1)\,l}{n}$ and $a' > \frac{l}{2}$, $a'' > \frac{2\,l}{3}$, &c. The general integral of (9) is

$$(10)\quad \frac{n\,y^{n-1}}{l^{n-1}} - \frac{n\,(n-1)\,(2\,y-l)^{n-1}}{2\,!\,l^{n-1}} + \frac{n\,(n-1)\,(n-2)\,(3y-2l)^{n-1}}{3\,!\,l^{n-1}} - \text{\&c.},$$

which vanishes for any value of $n$ when $y$ is taken at its lowest limits, but when $y$ is taken at its highest limit, $\frac{(n-1)\,l}{n}$, becomes

$$n\left(\frac{n-1}{n}\right)^{n-1} - \frac{n\,(n-1)}{2\,!}\left(\frac{n-2}{n}\right)^{n-1} + \frac{n\,(n-1)\,(n-2)}{3\,!}\left(\frac{n-3}{n}\right)^{n-1} - \text{\&c.},$$

$= 1$, for any value of $n$.

It is easy to see that on a straight line, not returning into itself, the number of possible positions for $n$ points is

$$(11)\qquad \frac{p^n}{n!} = \frac{l^n}{n\,!\,d\,l^n},$$

and the number of possible positions in which the extreme points are $x$ apart is $\frac{l-x}{d\,l} \times$ (number of possible positions on $x$ for $(n-2)$ points) $= \frac{(l-x)\,x^{n-2}}{(n-2)\,!\,d\,l^{n-1}}$; which divided by (11) gives

$$\frac{n\,(n-1)\,(l-x)\,x^{n-2}\,dl}{l^n},$$

as the *probability* that all the points are just included in $x$. This may be integrated between any limits less than $l$, and thus gives

$$(12)\quad \int_a^b \frac{n\,(n-1)\,(l-x)\,x^{n-2}\,d\,x}{l^n} = \frac{n\,(b^{n-1} - a^{n-1})}{l^{n-1}} - \frac{(n-1)\,(b^n - a^n)}{l^n},$$

as the *probability* that all the points are included in a length greater than $a$ and less than $b$.

Formulæ (6), (10), and (12), are sufficient to solve all the cases presented in the October Number of the Monthly. The decimal in Prop. 1, equal to $1 - \sqrt{\frac{1}{2}}$, and that in Prop. 7, equal to $\frac{1}{3}\,(1 - \sqrt{\frac{1}{2}})$ may be found by putting (12) and (6) equal to $\frac{1}{2}$, and solving with reference to $b$, $a$ being put equal to 0.

## AN ACCOUNT OF THE COMET OF DONATI. 1858.

### PART II.

For the following details, relating to the appearance presented by the comet in the telescope or to the naked eye, use has been made principally of the manuscript records of the Observatory of Harvard College, the results of observations made elsewhere not being accessible through the ordinary channels of information at the time of writing.

There was a marked increase of brilliancy accompanied by an equally perceptible lengthening of the tail, between the 10th and the 25th of September. Its sudden advance in size and splendor during the week following the latter date, was in perfect keeping with the often repeated history of bodies of its class. Every condition seemed to favor a rapid development. It was approaching the sun, which not only subjected it to a more intense illumination, but probably to other influences of a nature not well understood, but perfectly obvious in their effects; it was drawing nearer also to the earth, and was, besides, every night taking a more favorable position above the horizon, and presenting to better advantage the length and curvature of its train; lastly, the absence of moonlight towards the end of the month contributed more than any other circumstance to enhance the grandeur of the scene. Scarcely less impressive was the sudden vanishing of the spectacle a few weeks later, attributable to similar influences as it were reversed; — the comet receding from the sun and earth, contracting its dimensions, descending low in the mists of the horizon, and finally almost extinguished by the returning moonlight.

On the 8th, the diameter of the nucleus was ascertained to be two thousand miles. In immediate contact with it, was an intensely brilliant nebulosity, having a diameter of about three thousand

miles, while the surrounding diffused light extended forty or fifty thousand towards the sun. Measured by ordinary standards, this latter distance appears large, but it was manifestly insignificant compared with the effusion of nebulosity in the direction of the tail. Indeed the comparative absence of any considerable collection of diffuse light on the side nearest the sun, outside of the above radius, was so noticeable as to excite remark on several subsequent occasions. The fact of the position of the nucleus, precisely in the vertex of the train, must have been generally noticed. At this date the tail had acquired a length of sixteen millions of miles.

To ascertain the true dimensions of the nucleus, and to compare the intensity of its light with that of a star of equal brightness, the comet on the morning of the 9th was kept in the field of the great refractor by the clock-work motion, and the effect of the approach of daylight upon it, carefully noted.

In the early twilight the nucleus resembled a star of the fifth magnitude, subtending an angle corresponding to a diameter of five thousand miles; but owing partly to atmospheric disturbances, and partly to the difficulty of distinguishing its precise border, this proved to be much too large, for it diminished to less than half that amount when the daylight had become sufficiently strong to obliterate all but the true centre, which continued in sight until twelve minutes before sunrise; the light however no longer retained the scintillating, star-like character which distinguished it when seen on the background of a dark sky. At this time, therefore, the nucleus must have been nearly of the size of our moon, and probably shone with somewhat inferior intrinsic brightness.

Sept. 12. "A rapid increase in brightness, and length of train, the latter covers an arc of 6°."

"The intensity as well as the quantity of light emanating from the nucleus are the most distinctive features. To the naked eye, aided by the light of the envelope and contiguous part of the tail, it

was as bright as a star of the third magnitude. In the telescope the light concentrated within a circle of 10″ diameter, or six thousand miles, resembles that of a star of the fifth or sixth magnitude diffused over an equal space." The view of the comet on the morning of the 13th was still more satisfactory, owing to its greater elevation and the absence of moonlight.

To the naked eye on the 17th, the head equalled a star of the second magnitude. Its southern side (on the left hand and below as seen in the evening) was decidedly the brightest. A similar contrast was noticeable through a considerable extent of the tail for several weeks following; the convex outline being both brighter and more clearly defined than the opposite side. Ultimately this distinction disappeared, or rather it was reversed; the change taking place gradually, and becoming most noticeable after the 6th of October.

The commencement of a most important epoch in the physical history of the comet, dates in our records from the 20th of September. It is probable that symptoms of approaching changes, faintly indicated, may have appeared somewhat earlier; they were not however noticed on the 17th and 18th, on both of which occasions the comet was observed, though particular attention was not then given to the condition of the nucleus.

On the evening of the 20th, the train at its origin was plainly bifurcated, issuing from the head in two unequal streams forming its two sides, and leaving between them a dark space behind the nucleus. Their outline was a curve resembling an arc of the parabola or hyperbola. The southern stream was so much the more brilliant of the two, that in strong twilight this alone would have been seen as a short tail inclined by 30° or more to the true axis. "Between the nucleus and the sun is interposed an obscure crescent-shaped outline within which the light is unequally distributed, and has a strangely confused chaotic look; the details are too undecided

for precise description. There is also an elongation of the nucleus, which is singularly brilliant, or perhaps a ray extending a few seconds from it on the following or upper side. The exact character of these phenomena could not be made out, but they seemed to indicate the presence of some internal disturbing force."

Fig. 3* is taken from a sketch made at the time. The exterior dotted line represents the outline of the head of the comet, *n* the nucleus with the position of the ray, and *a a′ a″* the place of the obscure band. The latter was so faintly marked that it might easily have been overlooked, or have passed for an illusion.

Fig. 3.

Sept. 23d. "A fine clear sky with the moon nearly full. To the naked eye the head of the comet is as bright as a star of the first magnitude, and the train, notwithstanding the moonlight, is 6° or 8° long. It is already a brilliant object half an hour after sunset. The telescopic view is most extraordinary. The nucleus has diminished in size, being now only 3″, or 1300 miles, in diameter. Its light is exceedingly intense, and somewhat more concentrated than on the 20th. Outside of it is a bright envelope with its vertex in the direction of the sun, and 15″, or 6400 miles, distant. This is bounded on its outer margin by a dark band. The boundary of a second and less brilliant envelope is distant at its vertex about 30″, or 12800 miles, from the nucleus, and is terminated by a similar dark arch,

* In the figures the head of the comet is inverted, so as to correspond with its appearance in the telescope.

outside of which, again, is an atmosphere of faint, diffused nebulosity rapidly shaded off. The outlines can be distinguished through an arc of 220° or more, reckoned from the nucleus, but they extend considerably further into the train on the following or bright side."


Fig. 4.

The form and situation of the envelopes and their relative degrees of brilliancy are represented in Fig. 4 by gradations in the character and strength of the lines in the engraving.* The scale to which this cut is drawn has not been ascertained from exact micrometer measurements, and is not to be implicitly relied on in comparing the dimensions of the envelopes at this date with others where the proper angles at several points were more carefully determined.

Fig. 5.

Sept. 24th. The train is 7° in length, and evidently curved as represented in Fig. 5, which gives its appearance as seen at evening in the north-west. The telescopic view showed a decidedly dark axis to the tail, extending close up to the nucleus, which was elongated in a direction perpendicular to the axis. The inner envelope south of the nucleus was in part separated from it by a darker space and was twice as bright as the one

* The parallel curved lines in the wood-cut are employed merely to indicate the

next outside of it, which in turn was much brighter than the exterior nebulosity.

It is possible that the envelope, of which the outer edge is indicated in Fig. 6 by the letters $a\ a'\ a''$, corresponds to the bright nebulosity comprised between $a\ a'\ a''$, and $n$ in Fig. 3, and that the outline $b\ b'\ b''$ was formed between the 20th and the 23d. The dimensions of the two envelopes on the 24th, were found to be as follows. The letters refer to Fig. 6.

Fig. 6.

$n\,b' = 5800$ miles, $\qquad n\,a' = 13400$ miles,

$b\,b'' = 17000$ " $\qquad a\,a'' = 31000$ "

The obscure bands on the margins of the envelopes were 3″. 5 or 1400 miles broad. The estimated angle of divergence of the two forks of the tail at a distance of 10′ from the nucleus was from 20° to 25°.

On the 25th, the nucleus presented itself under a new aspect, in the act, as it afterwards proved, of disengaging a new envelope, or rather in a stage preparatory to that event. This perhaps is the first instance where an envelope has been seen in embryo at the surface of the nucleus, and has been traced through successive stages to a full development. The same phenomenon was subsequently illustrated in the case of the present comet by several exhibitions of

places where the light was more or less intense, and must not be too literally interpreted. The general effect is better given in Fig. 1; but all attempts at a faithful representation by means of an engraving, must, from the necessity of the case, be inadequate.

a similar nature; their history has a peculiar value, because it affords an insight into the mysterious processes by which the train is thrown out from the nucleus, under the stimulating influence of the sun's light and heat, or possibly of some unknown emanation from the same source. The following were the most conspicuous gradations of light recognized in its neighborhood. Commencing with the dark axis we have:—

1st. The axis, a narrow, well defined dark stripe penetrating quite up to the central body. Next in order towards the sun is the nucleus, on the eve, as we may say, of an eruption. The expression is fully warranted by its subsequent history. When seen to best advantage, two little streams of luminous matter were observed issuing from it one on each side, doubtless on their way to supply material to the tail now so rapidly expanding. Outside of the nucleus and of the nebulosity, apparently adhering to it, was a comparatively dark space, succeeded by the envelopes $b\,b'\,b''$ and $a\,a'\,a''$ with the intervening dark bands, and lastly, over the whole a thin veil of diffuse light; the latter attaining a distance of seventy thousand miles. If we include the dark axis, and the dark background of the sky, we have here nine alternations of light and shade, of various grades of intensity.

Fig. 7.

The micrometric measurements furnish the following dimensions, the letters referring to Fig. 7.

$$n\,b' = 7100 \text{ miles},$$
$$b\,b'' = 15600 \text{ “}$$

$n$ to dark space outside of $c' =$ 3500 miles. This space, first seen on the 24th, continued to enlarge until the envelope $b\,b'\,b''$ was completely severed from con-

nection with the nucleus. The breadth of the obscure band outside of $b\,b'\,b''$ was 1600 miles. At a distance of about three hundred thousand miles from its origin, the breadth of the tail was found to be one hundred and forty thousand miles. Its extreme length was 11°, and the breadth where widest 1°.

Sept. 27th. We have now conclusive evidence that the condition of the central luminosity on the 24th and 25th was that of an envelope in its earliest stages. "The outline of a new envelope is clearly distinguished. In form and position it is a miniature of that which has hitherto been the innermost. Like the latter it sets awry, inclining to the right hand side of the axis, as represented in Fig. 8. The outline of $a\,a'\,a''$ of Fig. 7 is becoming indistinct. The narrow dark stripe in the axis having its vertex precisely at the nucleus, is a remarkable object;" its width near its origin was found to be 1800 miles.

Fig. 8.

The tail had now attained a length of 13°, or eighteen millions of miles. A new appendage in the form of a long, narrow ray issuing from its convex side was seen as represented in Fig. 9, not following the curve of the tail proper, but projected nearly in a straight line from the sun. Its appearance simultaneously with the throwing off of a new envelope suggests the possibility of the two phenomena being in


Fig. 9.

some way connected with each other; both, it will be noticed, lie on the same side of the axis. Supposing it to have started from the head of the comet on the 25th, its velocity must have reached eight or ten millions of miles daily. Other comets have exhibited similar rays. That of 1843 shot out its streamers to a much greater distance. One which appeared in 1744 is said to have had no less than six, spread out like a fan.

On the 28th, the image of the nucleus in the focus of the large refractor afforded distinct photographic action, but the surrounding luminosity was not intense enough to form a picture. "The dark opening in the axis of the tail occupies about one twelfth of its breadth at a distance of 1° from the nucleus; it may be traced distinctly one or two degrees. The head of the comet, seen with a small telescope in strong twilight, which obliterates all but this brightest portion, is crescent-shaped as in Fig. 10. The tail is 19° long, or twenty-six millions of miles, with a streamer as on the 27th."


Fig. 10.

Sept. 29th. Between this date and the 30th, the comet passed its point of nearest approach to the sun, being distant about fifty millions of miles, and not quite seventy millions from the earth, which it was still rapidly approaching.

"Marked changes have occurred since the 27th. The little half moon envelope, then closely shrouding the nucleus, has elevated itself above it, and become the most conspicuous feature in the telescopic view. It is brightest near its outer edge," an indication that it was about to separate from the central nebulosity.

Fig. 11 will serve to convey an idea of the disposition of the envelopes; *a a′ a″* was fast losing its contour, which could with difficulty be made out; its place in the figure is not much to be

trusted. It will be remembered, that, within five days, all the nebulosity within the outline $c\,c'\,c''$ had been thrown off from the surface of the nucleus, rising from it at the rate of a thousand miles daily.

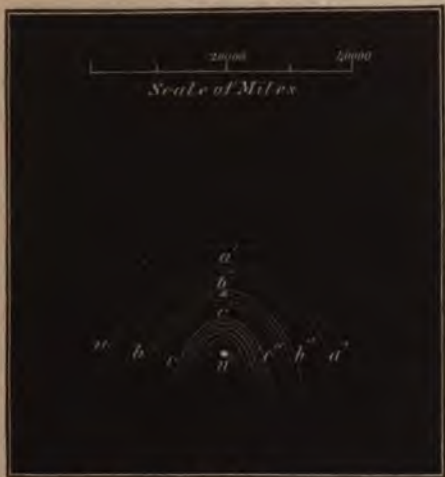


Fig. 11.

There is reason to suppose that the evolution was attended with something of violence, or of the nature of a sudden disruption, or of an explosion, if the expression does not convey too much the idea of motion apparent to the eye. There were rays or jets of light streaming in different directions from the centre, one in particular on the following (apparent right hand) side, imperfectly suggested on the 27th, now plainly seen, and there was a general aspect of confusion, suggesting the idea of internal disturbances. It was afterwards observed that as the nebulous matter rose higher and higher above its origin, it became uniformly blended, as if, when relieved from the immediate neighborhood of the nucleus, it was disposed to an even and symmetrical arrangement. The measured arcs gave the following results, $n$ in Fig. 11 designating as usual the nucleus, and $b'$ and $c'$ the vertices of the envelopes.

$$n\,b' = 10500 \text{ miles,}$$
$$n\,c' = \;\; 6000 \;\;\text{“}$$

The thickness of the brightest part of the arch under $b'$ was 2000 miles. Since the 24th, $b'$ has ascended (towards the sun) about five thousand miles, or only about one three thousandth part of the distance over which the end of the tail has advanced during the same interval.

Sept. 30th. “The edge of the envelope $c\,c'\,c''$ is very distinct,

and may be traced through an angle of 270°, reckoned from the nucleus. The latter is truncated, as it has often before been seen on the side opposite to the sun, giving it a half moon shape. The dark axis, which at its origin is almost black, and is of even breadth with the nucleus, completes the resemblance to a *phase* and *shadow*." There are objections to this explanation, although at first sight it is very plausible. Each new envelope as it emerges from the nucleus has the same phase-like form, while it is certainly everywhere permeated by the sunlight, a very small envelope still adhering to the nucleus would thus explain the peculiar form of the latter. The dark axis occupies a larger proportion of the whole breadth of the train at a distance of several degrees from the nucleus, than can with any probability be attributed to the defect of light intercepted by so small a body. It is moreover curved, which could not happen to a sensible amount in the shadow.

Perhaps two phenomena are here superimposed; a comparative deficiency of nebulosity towards the central regions of the tail, and an actual shadow perceptible a short distance only, close to the head of the comet, where at any rate we must assume the existence of a considerable collection of nebulous matter, sufficient to exhibit the outlines of a shadow cast upon it, if such really exists. This view receives some confirmation from a note of a later date. "The outlines of the axis-band are *straight lines* near the nucleus, but at a little distance they begin to blend with the general deficiency of light in the middle of the train." It seemed here to be conceivable that the shadow-margin and the outlines of the axis were distinct phenomena.

The tail to the naked eye was 22° long, or twenty-six millions of miles, and from 2° to 3° broad near its extremity, where also its rate of curvature was pretty suddenly increased. The upper outline was throughout brightest and best defined.

Oct. 2d. No new envelope had yet been formed, nor were any

PLATE I.

Drawn by G. P. Bond. Engraved by J. W. Watts.

Comet of Donati Oct. 2nd 1858.

indications of its approach manifested, although they were carefully looked for in the expectation that one would shortly appear. The nucleus, however, was unusually bright, and rounded on the side toward the sun. An increase of brilliancy in the nucleus was afterwards recognized as the precursor of a fresh eruption from its surface. Its diameter, perpendicular to the axis, was found to be less than 1600 miles.

There were three dark openings in the innermost envelope, between which it was intersected with bright rays. In Plate I., the engraver has given an eminently successful representation of the comet as it appeared in the field of the great refractor. The character of the light of the nebulosity composing the envelopes, and the appearance of the dark axial stripe penetrating, with well-defined outlines quite up to the nucleus, have been preserved with great fidelity.

Fig. 12.

The long narrow ray first noticed on Sept. 25th, springing from the convex side of the tail, was seen on the 2d of October as represented in Fig. 12.

The dimensions of the envelopes were as follows: —

| | |
|---|---|
| $a a' = 13200$ miles, | $n c' = 7500$ miles, |
| $b b'' = 3300$ " | $c c'' = 18900$ " |

The breadth of the brightest part of the tail, at a distance of [illegible]000 miles from the nucleus, was 90000 miles and its extreme length 25° to 30°.

The next date of observation was the 4th. Another envelope

PLATE

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The long narrow ray first noticed on Sept. 25th, springing from the convex side of the tail, was seen on the 2d of October as represented in Fig. 12.

The dimensions of the envelopes were as follows: —

$n\,b' = 13200$ miles, $\quad n\,c' = 7500$ miles,

$b\,b'' = 3300$ " $\quad c\,c'' = 18900$ "

The breadth of the brightest part of the tail, at a distance of 144000 miles from the nucleus, was 90000 miles, and its extreme length 25° to 30°.

The next date of observation was the 4th. Another envelope

was then rising, having already attained a diameter of above nine thousand miles within forty-eight hours. Its peculiar form and the position of a dark spot, *S*, are given in Fig. 13. A dark space now separated the envelope $c\,c'\,c''$ from the new one. In Fig. 13, we have from the micrometer measurements,
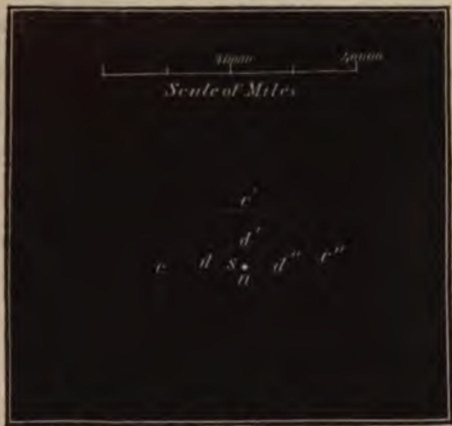
Fig. 13.

$$n\,c' = 8900 \text{ miles,}$$
$$n\,d' = 3050 \text{ "}$$
$$n\,S = 1800 \text{ "}$$
$$c\,c'' = 23500 \text{ "}$$
$$d\,d' = 9300 \text{ "}$$

The nucleus was smaller and less bright than on the 2d. The secondary tail was 35°, or thirty-four millions of miles long. On the 5th of October, the comet attained its greatest brilliancy. Its head was close to Arcturus, a star of the first magnitude, to which it was but little inferior in brightness, although the contrast in the *intensity* of their light was very evident. In Europe the two must have been seen still nearer to each other than they were in America, the nucleus passing a little to the south of the star, and the brightest part of the tail over it. The extremity of the train reached over Benetnasch and Mizar, the two southernmost stars in the tail of the Great Bear. It could be traced through an arc of 35°. Its breadth was 5° or 6°. With a little attention two additional streamers could be seen, one of


Fig. 14.

which was between 50° and 60° long, or above fifty millions of miles, with a slight curvature as in Fig. 14.

The interest of the telescopic view, taking all the circumstances into account, the size of the instrument, the perfect purity of the atmosphere, and the splendor of the object, have rarely been surpassed. The nucleus and the outline of its nearest envelope were visible in full sunshine with the large telescope. The head of the comet could be seen with the naked eye at twenty minutes after sunset, at which time the second envelope was discernible with the telescope. It is most remarkable, that, with all this accession of brightness, the nucleus itself had now diminished to a diameter of only four or five hundred miles, scarcely one fifth of what it was on the morning of the 9th of September, by a very careful determination. Its volume had thus diminished to *one twentieth* part only. The remaining nineteen twentieths had, in the intervening period, expanded into the tail, or had gone to form the envelopes which now encircled it, by a process which has been fully illustrated in the preceding pages. But are we then to conclude that the nucleus, the focus of these mysterious operations, had in this way expended the greater part of its substance? To this inquiry the best reply is a consideration of its subsequent condition. After several more eruptions from its surface, similar to those above described, it receded from our view about the 20th of October, with an evident *increase* of size compared with its condition two weeks before, and still shining with its accustomed intensity.

Examined in the daytime on the 5th with the highest powers which it would bear, no indication of a *phase* could be seen. The dark spot at *S*, of Fig. 13, had expanded in about the same proportion with the whole envelope in which it was situated. From near the vertex, and from the sides of the latter, there seemed to be an escape of jets of luminous gas, which streamed off like light spray thrown up against an opposing wind and driven before it.

The dimensions of the envelopes were as follows: —

$$n\,d' = 4200 \text{ miles}, \qquad n\,c' = 9500 \text{ miles},$$
$$d\,d'' = 8900 \text{ “} \qquad c\,c'' = 27000 \text{ “}$$
$$n\,b' = 14200 \text{ miles},$$
$$b\,b'' = 40500 \text{ “}$$

The diameter $d\,d''$ passed considerably above the nucleus. The outline $a\,a'\,a''$ of previous figures has faded away, and it is uncertain whether $b\,b'\,b''$ is the margin of the envelope so designated, or only that of the pretty sudden terminus of the stronger light comprising $a\,a'\,a''$ as well. The extreme diffusion of the light at some of the points measured, occasions a good deal of uncertainty in the above numbers; the distances to the vertices are usually the most trustworthy.

It will not be necessary to enter into the details of the history of other envelopes further than to indicate some of their leading features. Between the 2d and the 20th of October inclusive, four of them rose in succession from the nucleus. One, which was first seen on the 4th as just described, one between the 8th and 9th, another on the 15th, and a fourth on the 20th. The outlines of the brighter parts of three of them are shown in figures 15, 16, 17, 18. Plate II. has been engraved from a drawing of the comet selected from a series


Fig. 15.


Fig. 16.

executed with great care by Mr. Henry G. Fette. It gives a view of the region surrounding the nucleus on the evening of Oct. 10, when

[illegible] of the envelopes [illegible]

$[illegible] = 4200$ miles, [illegible] miles,
$dd'' = [illegible]$ " [illegible] "

$$ab' = 14200 \text{ [illegible]}$$
$$bb'' = 40500 \text{ [illegible]}$$

The diameter [illegible] passed [illegible] the [illegible]. The [illegible] and it is [illegible] whether [illegible] of the envelope [illegible] of the strongest light [illegible] of the light [illegible] of uncertainty [illegible] and usually [illegible].

[illegible] details of the history of [illegible] some of [illegible] October inclusive, [illegible]. One, which was first seen [illegible] the 8th and 9th, [illegible]. The outlines of the brighter [illegible] in figures 15, 16, 17, 18. Plate II. [illegible] comet selected from a series

Fig. [illegible]

Fig. [illegible]

executed with great care by Mr. [illegible] G. [illegible]. It gives a view of the region surrounding the nucleus on the evening of Oct. 10, when

PLATE II

*Comet of Donati Oct 10th 1858.*

it was at its least distance from the earth, and as it appeared in the field of the great refractor of the Observatory of Harvard College.

Fig. 15 is the outline on Oct. 6th of the envelope which made its appearance between the 2d and 4th; the places are indicated where it was intersected by brighter rays; its interior structure was very irregular. In Fig. 16, we have a representation of the inner envelope on Oct. 11th, the brighter portions only being included. Fig. 17 gives the outlines of the envelope first seen on Oct. 15th. Fig. 18 shows it at a more advanced stage, three days later.

Fig. 17.

Fig. 18.

A change, in the relative proportion of light distributed on the two sides of the principal axis of the comet, had been progressing up to about the 6th of October. At this date, although there may still have been a little more light on the right hand side, the difference was not nearly so large as it had been. The diameter of the nucleus was then 800 miles. On the 8th, its diameter was 1100 miles. The envelopes were most distinct on the left hand, or preceding side. The change was a permanent one, and for the future this became the brightest half of the head of the comet. It is curious to observe a corresponding change in the inclination of the envelopes to the axis. They now inclined even more decidedly to the left hand than they had at first done to the opposite side. The two last, those of Oct. 15th and 20th, seem in fact in the first instance to have issued as luminous jets or streams from the side, rather than from the

vertex of the nucleus. A similar reversal in the order of brightness was evident in the part of the train near the nucleus.

The tenth was the day of nearest approach to the earth, but the comet was manifestly on the wane, though expanded over a larger extent of the sky than before. Five envelopes, reckoning the exterior haze as one, could be traced through the whole or some part of their outline. The dark stripe of the axis was becoming less conspicuous, the central regions of the train being occupied with diffused light; on the 11th it was barely discernible. The last of the envelopes was thrown off on the 20th. The comet had now passed far to the south, and its low altitude prevented the continuance of the observations.

We must add a few words on the appearance presented by the tail between the 6th and the 10th of October. At the date first named, one of the supplementary rays, Fig. 19, attained a distance of 55°, or fifty millions of miles from the nucleus, somewhat exceeding that of the principal tail, and in a direction, as usual, nearly in a line from the sun. Others less perfectly developed could be discerned near a point where the curvature of the main stream was pretty suddenly changed. On the 8th, Fig. 20, five or six transverse bands could be distinguished in the tail half a degree or less in breadth, with clear, well-defined outlines, and perfectly resembling auroral streamers, excepting that they kept their position permanently, that is, without motion

Fig. 19.

sensible to the eye, they diverged from a point between the sun and the nucleus. The supplementary ray was not inserted in the original drawing from which Fig. 20 was engraved. Its place in the cut has been supplied from sketches on the 9th, and dates previous, allowing for its motion in the interval.

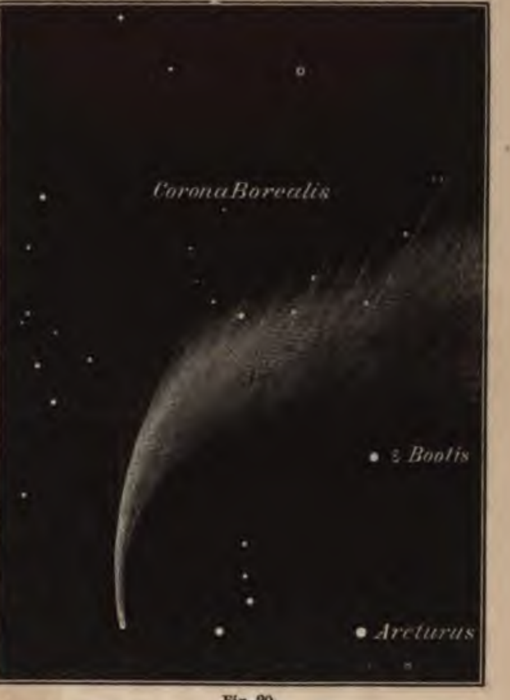

Fig. 20.

The train attained its largest apparent dimensions on the 10th, when the main stream of light could be distinguished through an arc of 60°, corresponding to a length of fifty-one millions of miles, or rather more than half the distance of our earth from the sun. The distribution of its light at a distance of 20° or 30° from the nucleus in parallel or slightly diverging bands, alternating with dark spaces, was strongly exhibited. They were 5° long, and 20′ or 30′ wide, and might aptly be compared either to the streamers which often break up the continuity of an auroral arch, or to a collection of five or six tails of small comets, forming from the remains of the large one. Whatever may have been their real nature, the impression to the eye involuntarily suggested the comparison. These bands were visible for one or two succeeding evenings, but were soon overpowered by the moonlight.

We will conclude with a review of some particulars relating to the comet which seem to deserve special attention. The dimensions

of the tail, and of the nucleus and envelopes on the several dates of observation, are given below. Apparent variations in the size of the nucleus were sometimes caused by disturbances in our own atmosphere, but in most cases the changes were undoubtedly real ones. The presence of moonlight, or of the slightest haze in the sky, had a very perceptible effect in diminishing the arc through which the tail could be traced. This will sufficiently explain the irregularities noticed in comparing its proportions from night to night.

| Date. | | | Length of tail | Breadth at extremity. | Remarks. |
|---|---|---|---|---|---|
| 1858. | Aug. | 29 | 2° = 14,000,000 miles. | | |
| " | Sept. | 8th and 9th | 4 = 16,000,000 " | | |
| " | " | 12 | 6 = 19,000,000 " | | |
| " | " | 17 | 4 = 10,000,000 " | | Moonlight. |
| " | " | 23 | 7 = 12,000,000 " | | " |
| " | " | 24 | 7 = 12,000,000 " | | " |
| " | " | 25 | 11 = 17,000,000 " | 1,500,000 miles. | |
| " | " | 27 | 13 = 18,000,000 " | | |
| " | " | 28 | 19 = 26,000,000 " | | |
| " | " | 30 | 22 = 26,000,000 " | 3,000,000 " | |
| " | Oct. | 2 | 25 = 27,000,000 " | 5,000,000 " | |
| " | " | 5 | 35 = 33,000,000 " | 5,000,000 " | |
| " | " | 6 | 50 = 45,000,000 " | | |
| " | " | 8 | 50 = 43,000,000 " | 7,000,000 " | |
| " | " | 10 | 60 = 51,000,000 " | 10,000,000 " | |
| " | " | 12 | 45 = 39,000,000 " | | |
| " | " | 15 | 15 = 14,000,000 " | | Moonlight. |

| Date. | | | Length of 'Streamers.' | Breadth at extremity. |
|---|---|---|---|---|
| 1858. | Oct. | 4 | 35° = 34,000,000 miles. | 1,000,000 miles. |
| " | " | 5 | 55 = 53,000,000 " | " " |
| " | " | 6 | 55 = 50,000,000 " | " " |

In computing the above, the curvature has not been regarded.

It must be borne in mind that we have taken for the extremity of the tail, the furthest point at which it was possible to detect a trace of it. It would scarcely have been noticed beyond 30° or 35°, even between the 5th and 10th of October, without a particular effort of the attention, and some training of the eye. The streamers, or additional rays, might easily have escaped notice altogether from their faintness. In making a comparison of the size of this comet with others, it will be best to limit the extent of the tail to the arc over which it was plainly visible, which would give a length of about thirty-five millions of miles. The shortening of the tail between the 12th and the 17th of September, is due entirely to the effect of moonlight. The more abrupt change between the 10th and 15th of October is partly due to the same cause, but there must also have been a great diminution in brilliancy.

For the nucleus we have the following measured diameters: —

1858, July 19. Diameter 5″ = 5600 miles. This probably includes the dense nebulosity immediately surrounding it, not distinguishable at the time from the true centre, on account of the low altitude of the comet.

Aug. 19. "Nucleus equals a star of the seventh magnitude."

" 29. Head of the comet visible to the unassisted eye as a star of the sixth magnitude.

Aug. 30. Diameter 6″ = 4660 miles. This result perhaps includes more than the true nucleus.

Sept. 8, 9. Diameter 3″ = 1980 miles. Taken just before sunrise, when all of the comet, excepting the nebulosity next outside, which was 3300 miles in diameter, was obliterated. On a dark sky the apparent diameter was 5280 miles, and the light equivalent to that of a star of the fifth magnitude.

Sept. 12. To the naked eye the head of the comet appeared as a star of the third magnitude.

On Sept. 17th, it equalled a star of the second magnitude.

Sept. 23. "To the naked eye the head of the comet is brighter than a star of the first magnitude." Its brilliancy at this date (one week before its perihelion passage, and seventeen days before its nearest approach to the earth), had reached a maximum. It is interesting to remark, that between the 17th and 23d was first noticed the characteristic formation of envelopes, which plainly operated as a check upon the accumulation of brightness at the central point. The nucleus, during the remaining period of its visibility, went through a series of periodic changes, acquiring more light just before an eruption, and suddenly diminishing after it. The variations, although evident to the eye, could not be accurately measured on account of the smallness of the angle subtended, and its want of precise definition.

On the 23d, its diameter, which appeared to be less than usual, was $3'' = 1280$ miles.

Sept. 24. Diameter $2''.5 = 1030$ miles.

Oct. 2. Diameter $5''.2 = 1560$ "

" 4. Nucleus evidently smaller than on the 2d.

" 5. Diameter $1''.5 = 400$ miles; "it is certainly less than $2'' = 540$ miles." This determination was made under most favorable conditions.

Oct. 6. Diameter $3'' = 800$ miles. The head of the comet nearly equalled Arcturus.

Oct. 8. Diameter $4''.4 = 1120$ miles. "The nucleus is decidedly brighter than on the 6th, and is preparing to throw off a new envelope."

Oct. 9. The nucleus had diminished in size simultaneously with the appearance of a new envelope.

Oct. 10. Diameter $2''.5 = 630$ miles.

" 11. Diameter $2'' \;\; = 510$ "

" 15. The head of the comet was as bright to the naked eye as a star of the third magnitude.

Oct. 18. Diameter $3'' = 900$ miles.

" 19. Diameter $3 \;\; = 920$ miles. The nucleus was compared with three stars of the sixth magnitude at the same altitude, and found to be far brighter than either of them. It was probably at least as bright as a star of the fifth magnitude, while to the naked eye the head nearly equalled one of the third magnitude.

Oct. 20. Diameter $2'' = 660$ miles. A new envelope was forming.

As before remarked, the least observed diameter of the nucleus, 400 miles, occurred on October 5th, the evening when the comet reached its maximum of brightness.

In order to exhibit the progressive motion of the envelopes from their point of origin, we give below in one view the distances of their vertices from the nucleus at different dates. The distances were measured in the line from the nucleus towards the sun. The better to distinguish them, we will use the following notation, which is the same with that employed in the wood-cuts.

| | | | | | | |
|---|---|---|---|---|---|---|
| $a' =$ | Vertex | of envelope | first seen | on | Sept. | 20. |
| $b' =$ | " | " | " | | " | 23. |
| $c' =$ | " | " | " | | " | 27. |
| $d' =$ | " | " | " | | Oct. | 4. |
| $e' =$ | " | " | " | | " | 9. |
| $f' =$ | " | " | " | | " | 15. |
| $g' =$ | " | " | " | | " | 20. |

Distances in miles from the nucleus ($n$) to the vertices of the envelopes $a$, $b$, and $c$: —

| | na′ | nb′ | nc′ | nd′ | ne′ | nf′ |
|---|---|---|---|---|---|---|
| Sept. 23 | *13000 | *6400 | | | | |
| " 24 | 13400 | 5800 | | | | |
| " 25 | *18000 | 7100 | | | | |
| " 27 | | 8400 | 3500 | | | |
| " 29 | | 10500 | 6000 | | | |
| Oct. 2 | | 13200 | 7500 | | | |
| " 4 | | | 8900 | 3050 | | |
| " 5 | | 14200 | 9550 | 4210 | | |
| " 6 | | | 10100 | 4270 | | |
| " 8 | | | 12400 | 7160 | | |
| " 9 | | | 13200 | 8650 | 1910 | |
| " 10 | | | 14100 | 8780 | 2760 | |
| " 11 | | | | *10200 | *4200 | |
| " 15 | | | | *11400 | 8160 | 3200 |
| " 18 | | | | *14500 | 9950 | 4400 |
| " 19 | | | | | 11200 | 5500 |

The vertex $g'$ had barely left the surface of the nucleus on the 20th.

The comet of Donati, although surpassed by many others in size, has not often been equalled in the intensity of the light of the nucleus. The diameter of the surrounding nebulosity on the other hand was unusually small, never much exceeding one hundred thousand miles, while that of the great comet of 1811, was ten times larger — its envelope attaining an elevation of more than three hundred thousand miles above the central body, exceeding by more than twenty times the largest of our measurements given above. Still it would be difficult to instance any one of its predecessors which has combined so many attractive features.

Its early discovery enabled astronomers, while it was yet scarcely distinguishable even with the telescope, to predict, some months

* The numbers marked with an asterisk are less reliable than the others.

in advance, the more prominent particulars of its approaching apparition, which was thus observed with all the advantage of previous preparation and anticipation. The perihelion passage occurred at a most favorable moment for presenting the comet to good advantage. When nearest the earth, the direction of the tail was nearly perpendicular to the line of vision, as represented in Fig. 21, so that its proportions were seen without foreshortening. The position of the comet relatively to the sun and to the earth at the date of discovery, June 2d, and when nearest the earth, Oct. 10th, will be understood from the figure. Its situation in the latter part of its course afforded also a fair sight of the curvature of the train, which seems to have been exhibited with unusual distinctness, contributing greatly to the impressive effect of a full length view. Frequent allusion has been made to the influence of the light of the moon on the visibility of the comet. Few readers will be aware how much of its splendor and vast dimensions, during the first ten days of October, we owe to the fortunate circumstance that, at this critical period, the moon was absent from our evening skies. The effect of the presence of a full moon, though simply optical, and due only to the force of contrast, would have been quite as prejudicial as if the comet had lost two thirds of its train, and as large a proportion of the brightness of the remaining third; above all, we must have lost those most singular phenomena — the supplementary rays, and the alternating bright and dark bands in the train; the latter seem to have been new in cometary history. Supposing the substance of the tail to be driven off into space, never again to return to its orig-

Fig. 21.

inal source, the inquiry at once arises, What then becomes of it? The appearances in question show plainly enough a process of separation into distinct masses, and in each of these a tendency to condense about a central axis.

It is remarkable that the aggregation should have been around separate axes, rather than about one or more central points, and that the axes should have manifested a disposition to diverge from the sun; as though these collections of nebulosity were in reality a group of new comets in process of formation. The increasing moonlight and low altitude of the comet would not allow their being followed to a more complete development.

The condition of the nucleus and neighboring region has received a large share of attention in the preceding pages, because it has afforded so ample an illustration of phenomena of which, up to the present time, very little has been certainly known. The comets of 1744 and of 1811 had well-formed envelopes, but the observations upon them were too imperfect and disconnected to afford much more than a basis for conjecture as to their origin and destination. That of HALLEY, at its apparition in 1835–36, furnishes an example more nearly parallel to the present one, but its phenomena were on a comparatively feeble scale.

The most recent intelligence leaves no room to doubt that the comet of DONATI is periodical, having a time of revolution of about two thousand years. The following are the results arrived at by different computers: —

| | | |
|---|---|---|
| WATSON, | 2415 | years. |
| BRUHNS, | 2102 | " |
| LÖWY, | 2495 | " |
| GRAHAM, | 1620 | " |
| BRÜNNOW, | 2470 | " |
| NEWCOMB, | 1854 | " |

The last two determinations are based upon longer intervals of observation than the others, Mr. Newcomb's being a few days longer than that of Dr. Brünnow. The remaining uncertainty in the period will be materially reduced, when observations have been received from the southern hemisphere, where the comet is still in sight.

The subjoined table contains the distances of the comet from the sun, and from the earth, and its hourly rate of motion:—

| 1858. | Distance from Sun in miles. | Distance from Earth in miles. | Hourly Velocity in miles. |
|---|---|---|---|
| June 2, | 215,000,000 | 240,000,000 | 65,000 |
| July 2, | 173,000,000 | 240,000,000 | 72,000 |
| Aug. 2, | 127,000,000 | 220,000,000 | 84,000 |
| Sept. 1, | 82,000,000 | 160,000,000 | 105,000 |
| " 11, | 70,000,000 | 130,000,000 | 115,000 |
| " 21, | 60,000,000 | 95,000,000 | 124,000 |
| Oct. 1, | 56,000,000 | 66,000,000 | 128,000 |
| " 11, | 61,000,000 | 52,000,000 | 123,000 |
| " 21, | 71.000,000 | 67,000,000 | 114,000 |

Supposing its last perihelion passage to have occurred at the beginning of the Christian era, it must have passed its aphelion in the early part of the tenth century, at a distance of 14300 millions of miles from the sun, its velocity at that point being 480 miles an hour.

# Mathematical Monthly Notices.

*An Elementary Treatise on Mechanics; designed as a text-book for the University Examinations for the ordinary degree of B. A. Part I. Statics. Part II. Dynamics of a Particle. By* J. B. CHERRIMAN, *M. A., late Fellow of St. John's College, Cambridge, and Professor of Natural Philosophy in University College, Toronto.* MACLEAR & Co., King street East. 1858.

THE author says: "As my design has been only to furnish to students a text-book for such parts of the subject as are required for the ordinary degree of B. A. in Universities, I have not thought it advisable to burden this work with mere explanation or illustration, or to add examples; presuming that such, where necessary, will be furnished in the lecture-room by the tutor."

We have read this work with sufficient care to be satisfied that it meets a want in the usual undergraduate course of instruction. Comparatively few students acquire sufficient knowledge of the higher mathematics, to be able to read with much advantage a work on mechanics, in which the notation and logic of the calculus is used; while the elementary mathematics should be far too thoroughly learned by all, not to fit the student to read something more rigorous on the subject than is found in our ordinary works on Natural Philosophy. We have here as full a work on Mechanics as the student will be likely to have time to accomplish during his college course, which demands a knowledge of only the simplest elements of Mathematics. Indeed, in many of our Academies and High Schools, the mathematical standard is sufficiently high to enable students to read this work without much difficulty, and with far greater profit than they usually derive from a merely descriptive or experimental course of Mechanics.

The work is well arranged, the definitions and demonstrations are clear and concise, and our only regret is, that the author did not think best to collect under each head a few carefully chosen examples.

Besides the fact, that every teacher who supplies this deficiency performs the labor which the author might have saved to all, we are afraid that far too many will want the time or disposition to supply it at all, and thus the student will fail to reap all the advantage he ought to derive from the study of the work.

*Journal de Mathématiques Pures et Appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques; Publié Par* JOSEPH LIOUVILLE, *Membre de l'Académie des Sciences et du Bureau des Longitudes, Professeur au Collége de France.* Paris: MALLET-BACHELIER, Quai des Augustines, No. 55.

For those who have been in the habit of consulting the pages of this valuable Journal during the twenty-three years of its existence, this brief notice is unnecessary; and we only insert it, for the information of the younger portion of our readers, as part of a plan in which we propose to give brief historical notices of all the Mathematical Journals at present existing, as well as of those which have been discontinued. Besides those which have existed in our own country, there are quite a number of discontinued journals in Europe of far too great value not to be generally known, and especially by those who have any interest in tracing the progress of the Mathematics since the days of Newton and Leibnitz.

The "Annales" of M. Gergonne, to which we shall hereafter refer more particularly, was discontinued in 1831, after a most successful issue of twenty years. In the "Avertissement" M. Liouville announces his Journal as the continuation of the "Annales," and adds, "Our Journal, like that of M. Gergonne, will be a monthly. The first number will appear in January, 1836, and the following ones from month to month, with all desirable punctuality. The numbers will be of unequal size, and vary from thirty-two to forty pages 4to, according to the nature of the memoirs which they contain. Together they will form a large volume each year, containing all the plates necessary for the understanding of the text."

How completely this prediction of the Editor has been fulfilled, the twenty-three volumes, completed with the issue of the number for December, 1858, averaging over 450 quarto pages per volume, sufficiently attest.

In the "Avertissement" to the twenty-first volume (1856), M. Liouville remarks: "I established this Journal at a time when the facilities of publication for young geometers were much less than at present. We had, it is true, Crelle's Journal and that of the Polytechnic School; but the *Correspondence* of M. Hachette, the *Annales* of M. Gergonne, the *Bulletin des Sciences* of M. Férussac, &c., had been discontinued for many years. Our first number was for January, 1836; and it was only six months after, at its second semester, that the Academy of Sciences conceived its *Comptes Rendes*, afterwards imitated by the majority of the Learned Societies. We have, since that time, issued a volume each year; the first twenty volumes composing the first series. The one we now offer to the public begins a new series."

The number for December, 1858, will complete the third volume of the new series. It will give only an indefinite idea of the contents of this Journal to add, that its first twenty-two volumes contain eight hundred Notes and Memoirs, in over ten thousand large quarto pages, comprising contributions from nearly all the eminent European mathematicians of the last quarter of a century.

# Editorial Items.

THOSE wishing to secure the services of a gentleman of large experience, eminently successful both as an author and teacher, and for many years Professor of Mathematics in one of the oldest and most influential Colleges in the country, will please apply to the Editor of the Mathematical Monthly for further information. . . . . . ASAPH HALL, Esq., Assistant at Harvard College Observatory, sends the following erratum: — Article 32, page 35 of DAVIS's Translation of GAUSS's "Theoria Motus," for $\tan \frac{1}{2} v = \frac{(u-1) \tan \frac{1}{2} \psi}{u+1}$ read $\tan \frac{1}{2} v = \frac{u-1}{(u+1) \tan \frac{1}{2} \psi}$. This error is also in the original. . . . . . In reply to numerous inquiries concerning publications of interest to mathematicians, issued in this country, we subjoin the following NAUTICAL ALMANAC PUBLICATIONS: Tables of the Moon, constructed from PLANA's Theory, with AIRY's and LONGSTRETH's corrections; HANSEN's two inequalities of long period arising from the action of Venus, and HANSEN's values of the secular variations of the mean motion, and of the motion of the perigee. Arranged for the use of the Nautical Almanac in a form designed by Professor BENJAMIN PEIRCE, Consulting Astronomer, under

the superintendence of CHARLES HENRY DAVIS, Lieutenant United States Navy; and published under the authority of the Hon. JOHN P. KENNEDY, Secretary of the Navy. Washington: 1853. Nautical Almanac and American Ephemeris for the years 1855 to 1860, inclusive; the volume for 1861 being nearly ready. The nautical part, for the years 1855 to 1861, inclusive, has been published separately, especially for the use of navigators. The sale of this edition has reached six thousand copies per annum. The volume of the American Ephemeris for 1857 contains Prof. CHAUVENET'S valuable methods for "Correcting Lunar Distances," and for "Finding the Error and Rate of a Chronometer by Equal Altitudes;" also "Logarithms of the Le Verrier Coefficients of the Perturbative Function of Planetary Motion." By the late SEARS C. WALKER. . . . . . In future numbers of the Monthly we shall give the titles of all Memoirs upon mathematical subjects ever published in this country. If authors, publishers, or societies publishing Memoirs, will in future send us the titles, in full, of works or memoirs in any departments of the Pure or Applied Mathematics, as soon as published, we will insert them in the Monthly. . . . . . It gives us pleasure to add the following names to our list of coöperators and contributors. M. L. COMSTOCK, Esq., Tutor in Knox College, Galesburg, Ill.; W. W. DICKSON, Esq., Pittsburg, Pa.; Prof. MARK D. HANOVER, Principal of Liberty Academy, Springfield, Tenn.; R. G. HATFIELD, Esq., Architect, N. Y.; EPHRAIM HUNT, Esq., Sub-master English High School, Boston; Prof. LEM. JOHNSON, Normal College, N. C.; C. J. KEMPER, Esq., Harrisburg, Va.; Prof. N. H. LOSEY, Knox College, Galesburg, Ill.; Prof. SAMUEL SCHOOLER, Edge Hill Academy, Guiney's P. O., Va.; JOHN B. THOMPSON, Esq., Charlottsville, Va.; Rev. A. D. WHEELER, Brunswick, Maine; G. T. Kingston, M. A., Director of the Magnetic Observatory, Toronto, Canada; Prof. WRAY BEATTIE, Iowa Wesleyan University, Mount Pleasant; Prof. J. T. BENEDICT, New York Free Academy; H. WILLEY, Esq., New Bedford, Mass.; JOHN W. BULKLEY, Esq., Superintendent of Schools, Brooklyn, New York. . . . . . BOOKS RECEIVED. Nouvelles Annales de Mathématiques for October, 1858. Paris: Mallet-Bachelier. Elements of Descriptive Geometry. Part I. The Point, the Straight Line, and the Plane. By SAMUEL SCHOOLER, M. A. J. W. Randolph, 121 Main street, Richmond, Va., 1853. We shall be much obliged if every institution of learning in the country, publishing a catalogue, will send a copy to the Mathematical Monthly. . . . . . We trust our friends will pardon any want of punctuality on our part, in answering letters, or acknowledging the receipt of communications. We should be happy to answer at once, in all cases; but this, with our other duties, is simply impossible; and, therefore, as a rule, we only reply in those cases, in which there seems to be an immediate necessity, and the reception of the following number of the Monthly will not be a sufficient reply. . . . . . The terms of the Monthly are strictly and invariably in advance, for the simple reason that it is absolutely necessary for us to know upon what we are to rely to meet the year's expenses; and we hope, therefore, to be able to send receipted bills in the January Number to those whose subscription now remains unpaid.

# THE
# MATHEMATICAL MONTHLY.

Vol. I...JANUARY, 1859....No. IV.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

If the triangle $DEF$ be inscribed in the triangle $ABC$, the circumferences of the circles circumscribed about the three triangles $AEF$, $BFD$, $CDE$, will pass through the same point.

### II.

Given the base of a spherical triangle, and the ratio of the tangents of the angles at the base; to find the locus of the vertex.

### III.

If in any triangle a line be drawn from the vertex of either angle to the opposite side, bisecting the angle, prove that the product of this line and the secant of half the bisected angle equals a harmonic mean between the two sides containing the bisected angle.

### IV.

If $A$, $B$, $C$, and $D$ be any four points in the same plane, so situated that four circles $ABC$, $ABD$, $ACD$, and $BCD$ can be drawn through them three and three, prove that the circumferences of any two of these circles will intersect at the same angle as the circumferences of the remaining two.

V.

A paraboloid of given dimensions, but unknown specific gravity, is immersed in common water, until its summit coincides with the surface of the fluid. The pressure from above being removed, the body ascends by the force of the water until its base coincides with the fluid's surface, and then descends, and so on. Find from this circumstance the specific gravity of the body.

The solutions of these problems must be received by the first of March, 1859.

## THE ORDER OF MATHEMATICAL STUDIES.

By Rev. Thomas Hill. Waltham, Mass.

The order in which Nature presents ideas to the infant mind, is the proper order in which those ideas should be systematically developed. And the first mathematical idea that enters a child's mind is that of form; the child recognizes a vast variety of objects by their form before it can count. Geometry is therefore the first mathematical study for a child, and should precede arithmetic.

But theorems and demonstrations are wholly unsuited for a child; geometry must be recognized as food for other powers than those of reasoning. Nature presents forms to the eye, and stimulates the child's conception of figures, years before it is capable even of the simplest process of geometrical reasoning. Geometry should, therefore, in a natural system of education, begin with addressing the eye, and stimulating the powers of observation. Little bricks, Chinese tangrams, rude compasses, blackboard drawings, and similar means of illustrating form and the laws of form to the eye, should be in constant use from an early age. The habit of exactness in laying the bricks and blocks, and of critically comparing

and analyzing figures drawn on the blackboard, early formed, will be a great aid in gaining that quickness and accuracy of observation which is one of the rarest and most valuable of intellectual powers. Number must also be first presented, as in nature, in the concrete form; and the proper time for doing this is evidently to be found at the period when the symmetrical building with bricks, or the symmetrical chalk drawings, have introduced the idea of number as a distinct intellectual element. But the first lessons in number should evidently be concrete, such as may be given by a pint of corn. From the actual separation of numbers into their equal parts by separating the corn into equal heaps, will come the first clear ideas of prime and composite numbers. In like manner the idea of multiplication, and the commutation of factors; of division, and the process of dividing by the quotient to find the divisor; of the rate of increase in powers; and of the rapidity with which numbers increase in decimal progression, can be clearly conveyed in no other way than by beans, counters, or corn.

The next step is to appeal to the imagination, and develop systematically the powers of conception. This is the peculiar office and excellence of geometry, and yet it is a point to which writers on that science have seldom referred. All mathematics, and indeed all studies and occupations of life, require the ability to conceive clearly as a real thing, that which has been described in words. The first study to require and develop this ability is found in simple geometry. But inasmuch as the powers of conception are developed much earlier than those of reasoning, it seems to me proper that a child should be taught to conceive of geometrical truths before it is taught to demonstrate them. They may be presented to him in a logically connected series, and in simple forms of language, not avoiding the scientific names of figures, but carefully avoiding scientific terms in the definition and description of the figure. A judi-

cious selection of geometrical facts and names may be thus stored in the child's memory while you are at the same time giving him a power of quick and accurate conception, which will enable him to solve all ordinary questions of loci at a single glance, without reasoning, but by direct sight. Nor in giving him facts should we confine ourselves to those which may be most readily demonstrated, but rather to those which will most stimulate the imagination, and which will lure him upward with a desire to demonstrate them.

The corresponding period in arithmetic introduces the child to the rules of written arithmetic. The decimal notation in Arabic figures is, of course, the first thing to be learned; and it should be taught, at first, as extending on either side of the unit's place. No advantages, on the contrary great disadvantages, arise from postponing a knowledge of this law on the right of the decimal point, to a later period. The moment that a child is able to understand the meaning of 345 he can also understand 3.45 or .345; and the postponement of an explanation of the latter expressions to a later period, invests them with factitious difficulties, that will impair the pupil's freedom in the use of decimals for many years, if not for life. Prof. Loomis, in his note (Math. Month. p. 73), seems not to have noticed the main point of my remark, and omits my careful qualification "at first" from the phrase on which he comments. At the age of fourteen or fifteen years a child has his reasoning powers somewhat developed, and will begin to relish the demonstration of both arithmetical and of geometrical problems. And herein also the course of nature should be followed. The first essays toward demonstration are usually by nature analytical, in the metaphysical sense of that word, and yet almost all writers on geometry make use almost wholly of synthesis. About the same period of the pupil's life he may begin algebra, at first as an extension of arithmetic, afterward as the law of all magnitude, and especially of unknown or variable elements.

After this period, the order of study becomes not unimportant, but less important, than for a younger child. The powers of observation, if not cultivated in early childhood, are apt to become permanently dulled; and the same is true of the powers of conception. Not only do I find in the primary schools in which geometry is studied, that the scholars of eight to ten years old are quicker in understanding it than those from twelve to fourteen; but I have noticed that the same individual, in passing from the younger to the older period without any cultivation of his geometrical tastes, has lost, in his power of understanding, my isolated experimental lessons. I have, therefore, thought it worth while to occupy thus much room in the pages of the Monthly, to call the attention of teachers to the importance of mathematical training in the earliest years; and more especially to the importance of restoring geometry to its ancient place as the foundation of learning.

## PROPOSITIONS RELATING TO THE CONE AND SPHERE.

BY E. S. SNELL.
Professor of Mathematics in Amherst College, Amherst, Mass.

There is a right cone, which appears to be as remarkable among cones, as the cylinder of Archimedes among cylinders. It is that whose slant height has to the radius of its base, and consequently its convex surface to the area of its base, the ratio of *three* to *one*. Some of its curious properties I presented about ten years ago in Silliman's Journal. Those, with two or three others since observed, I briefly state without proof.

1. The right cone, whose slant height is to the radius of base as 3 : 1, has the *greatest volume* within a given surface.

2. It has *less entire surface* than any other cone circumscribing the same sphere.

3. It has *less volume* than any other cone circumscribing the same sphere.

4. It has *twice* the *height* of the inscribed sphere.

5. Its *entire surface* is *twice* that of the inscribed sphere.

6. It has *twice* the *volume* of the inscribed sphere.

7. Its circle of contact on the inscribed sphere is so situated, that the surface of the segment *below* it, is *twice* that of the segment *above* it.

8. The half surface of the inscribed sphere is *twice* the area of the base of the cone.

## MAGIC SQUARE FOR THE YEAR 1859.

BY J. WIESSNER. Washington, D. C.

From 109 to 229.

1859.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 188 | 201 | 214 | 227 | 119 | 121 | 134 | 147 | 160 | 173 | 175 |
| 176 | 189 | 202 | 215 | 228 | 109 | 122 | 135 | 148 | 161 | 174 |
| 212 | 225 | 117 | 130 | 132 | 145 | 158 | 171 | 184 | 186 | 199 |
| 200 | 213 | 226 | 118 | 120 | 133 | 146 | 159 | 172 | 185 | 187 |
| 127 | 140 | 142 | 155 | 168 | 181 | 194 | 207 | 209 | 222 | 114 |
| 115 | 128 | 141 | 143 | 156 | 169 | 182 | 195 | 197 | 210 | 223 |
| 224 | 116 | 129 | 131 | 144 | 157 | 170 | 183 | 196 | 198 | 211 |
| 151 | 153 | 166 | 179 | 192 | 205 | 218 | 220 | 112 | 125 | 138 |
| 139 | 152 | 154 | 167 | 180 | 193 | 206 | 208 | 221 | 113 | 126 |
| 164 | 177 | 190 | 203 | 216 | 229 | 110 | 123 | 136 | 149 | 162 |
| 163 | 165 | 178 | 191 | 204 | 217 | 219 | 111 | 124 | 137 | 150 |

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## EXTRACTION OF THE HIGHER ROOTS OF NUMBERS.

BY DASCOM GREENE.
Professor of Mathematics in Rensselaer Polytechnic Institute, Troy, N. Y.

In a note on the extraction of the cube root of numbers in No. II. of the Mathematical Monthly, by the editor, an easy method is given of forming each trial divisor from the preceding complete divisor by the addition of numbers already used. Another simple method is as follows: From equations (4) and (5) of that article,

$$\text{1st complete divisor} = 3\,r_1^2 + 3\,r_1\,r_2 + r_2^2,$$

$$\text{2d trial divisor} \qquad = 3\,r_1^2 + 6\,r_1\,r_2 + 3\,r_2^2,$$

$$= 1\,(3\,r_1^2) + 2\,(3\,r_1\,r_2) + 3\,(r_2^2);$$

hence, from any complete divisor we find the next trial divisor by multiplying its first term by 1, the second by 2, the third by 3, and taking the sum.

An advantage of this method is, that it may be extended to the case of any higher root. Thus, $N$ being the given number, and supposing $r_1$ to have been already found, we shall have

$$N^{\frac{1}{n}} = r_1 + r_2;$$

or $$N = (r_1 + r_2)^n = r_1^n + n\,r_1^{n-1}\,r_2 + \frac{n\,(n-1)}{1\,.\,2}\,r_1^{n-2}\,r_2^2 + \&c.$$

(1) $$\therefore r^2 = \frac{N - r_1^n}{n\,r_1^{n-1} + \frac{n\,(n-1)}{1\,.\,2}\,r_1^{n-2}\,r_2 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,r_1^{n-3}\,r_2^2 + \&c.}$$

We now have

$$N^{\frac{1}{n}} = r_1 + r_2 + r_3;$$

or $$N = (r_1 + r_2 + r_3)^n$$
$$= (r_1 + r_2)^n + n\,(r_1 + r_2)^{n-1}\,r_3 + \frac{n\,(n-1)}{1\,.\,2}\,(r_1 + r_2)^{n-2}\,r_3^2 + \&c.$$

(2) $$\therefore r_3 = \frac{N - (r_1 + r_2)^n}{n\,(r_1 + r_2)^{n-1} + \frac{n\,(n-1)}{1\,.\,2}(r_1 + r_2)^{n-2}\,r_3 + \&c.}$$

The trial divisor for finding $r_3$ is

$$(3)\qquad n\,(r_1+r_2)^{n-1}=n\,r_1^{\,n-1}+n\,(n-1)\,r_1^{\,n-2}\,r_2 + \frac{n\,(n-1)\,(n-2)}{1.2}\,r_1^{\,n-3}\,r_2^2+\text{\&c.},$$

and its terms are evidently formed from those of the complete divisor in (1) by multiplying the first by 1, the second by 2, the third by 3, and so on.

As a first example, let us take that contained in the article above referred to; the operation is as follows.

```
162467446993496          | 54566 = cube root,
125          3 × 5^2 = 75.. | 75..
 37467 3 × 5 × 4 =  60.  | 120.
            4^2 =  16    |  48
 32464      = 4 × 8116 | 8748.. | 8748..
  5003446   3 × 54 × 5 =   810. |  1620.
                  5^2 =    25   |    75
  4414625         = 5 × 882925 | 891075.. | 891075..
   588821993       3 × 545 × 6 =   9810.  |  19620.
                          6^2 =      36   |    108
   535233816            = 6 × 89205636 | 89303808..
    53588177496          3 × 5456 × 6 =       98208.
                                6^2 =           36
    53588177496                 6 ×      8931362916
```

As a second example, let it be required to extract the 5th root of 2956466552832.

```
2956466552832 | 312 = 5th root.
243                5 × 3^4 = 405....   | 405....
 5264665     10 × 3^3 × 1 =   270...   |  540...
             10 × 3^2 × 1^2 =   90..   |  270..
              5 × 3 × 1^3 =     15.    |   60.
                      1^4 =      1     |    5
  4329151            = 1 × 4329151 | 4617605....
   93551452832       10 × 31^3 × 2 =    595820...
                     10 × 31^2 × 2^2 =   38440..
                      5 × 31 × 2^3 =      1240.
                               2^4 =        16
   93551452832              = 2 ×    46775726416
```

We propose hereafter to give an application of this method of extracting roots to the resolution of the higher numerical equations.

# INVESTIGATION OF THE NATURE OF THE CURVE OF A DRAWBRIDGE.

BY WILLIAM WATSON.
Tutor in the Lawrence Scientific School, Cambridge, Mass.

The following investigation was made to determine the equation of the curve which sustains the counterpoise of a drawbridge in such a manner, that the movable platform is in equilibrium in every possible position.

The annexed figure, which is drawn from a model recently presented to the Lawrence Scientific School, represents the bridge in elevation. $AB$ is a vertical projection of the platform, which rotates upon a horizontal axis, projected at $A$. $AC$ is a post, at the top of which there is a fixed pulley, $C$. $ICE$ is a chain, attached to the platform at $I$, passing over the fixed pulley $C$, and connected with a roller $E$, which rests upon a rigid curve, $CF$.


The problem then is, to determine the nature of the curve, such that the platform shall constantly be in a state of equilibrium.

Let $C$ be the origin of polar coördinates $(r\varphi)$; $CA$ the prime radius. Let $W$ be the weight of the platform; $W_1$ that of the roller. Let $c$ be the length of the chain; $l$ that of the platform. Let the angle $BAC$ be denoted by $\theta$; the angle of $r$ and the corresponding tangent by $\varepsilon$. Let $a$ be the height of the post, the same as $AI$. Assume $A$ as a centre of statical moments; and consider the curve smooth. The statical moment of the platform about $A$ is the product of its weight by its lever arm, that is, $\frac{1}{2}Wl\sin\theta$; that of the

roller is the product of $T$, the tension of the chain, by the arm $AL$; but since the tension $T$ acts to produce rotation in an opposite direction to that produced by $W$, the moment must be regarded as negative; that is, $-T \times AL$. But

$$T = W_1 \cos(\varphi + \varepsilon) \sec \varepsilon; \text{ and } AL = a \cos \tfrac{1}{2} \theta.$$

$$\therefore - T.AL = - W_1 \cos(\varphi + \varepsilon) \sec \varepsilon . a \cos \tfrac{1}{2} \theta$$

$$= \text{the statical moment of the roller.}$$

Since the bridge is in equilibrium in every position, the sum of the statical moments must be zero.

$$\therefore \tfrac{1}{2} W l \sin \theta - W_1 \cos(\varphi + \varepsilon) \sec \varepsilon \, a \cos \tfrac{1}{2} \theta = 0;$$

or, reducing, $Wl \sin \frac{1}{2} \theta - W_1 a (\cos\varphi - \sin\varphi \tan \varepsilon) = 0$.

Substituting for $\sin \frac{1}{2} \theta$ its value $\frac{c-r}{2a}$, we have

$$Wl\left(\frac{c-r}{2a}\right) - W_1 a (\cos\varphi - \sin\varphi \tan\varepsilon) = 0.$$

Substituting for $\tan \varepsilon$ its value $r D_r \varphi$, and $B$ for $\frac{2a^2 W_1}{Wl}$, we have

$$c - r - B(\cos\varphi - \sin\varphi \, r D_r \varphi) = 0,$$

or $$(c - r - B\cos\varphi) D_\phi r + B r \sin\varphi = 0.$$

The integral of this equation, since it satisfies the usual conditions of integrability,

$$D_\phi (c - r - B\cos\varphi) = D_r (B r \sin\varphi),$$

is $$\int_r (c - r - B\cos\varphi) + \Phi = \text{constant},$$

or $$cr - \frac{r^2}{2} - B r \cos\varphi + \Phi = \text{constant};$$

$\Phi$ being an arbitrary function of $\varphi$. To determine $\Phi$ we have

$$\Phi = \int_\phi \left[ D_\phi \left( cr - \frac{r^2}{2} - B r \cos\varphi \right) - B r \sin\varphi \right] = \int_\phi 0 = \text{constant}.$$

The most general equation of the curve is therefore

$$2cr - r^2 - 2Br\cos\varphi = \text{constant}.$$

Taking the origin on the curve, the equation reduces to

$$r = 2c\left(1 - \frac{B}{c}\cos\varphi\right).$$

If $B=c$, this equation becomes that of the cardioid referred to its cusp, in which the length of the chain $c$ becomes the radius of the generating, as well as that of the directing circle.

I propose, in some future number of the Monthly, to discuss the general equation.

## ON A SIMPLIFICATION IN COMPUTING EARTHWORK.

By J. B. Henck. Civil Engineer, Boston.

When ground of considerable extent has been divided, preparatory to excavation, into equal rectangles, it often happens, that the surface within some of these rectangles rises or falls so much towards the centre, as to make it necessary to take with the level the height at the centre, in addition to the usual heights at the corners. Figure 1 represents such a case. $F\,B\,C\,D\,E$ is the ground surface. $B, C, D, E$ are the corners where heights are taken, $F$ is the point where the additional height is taken, and $F_1\,B_1\,C_1\,D_1\,E_1$ is the bottom of the excavation. The mass of earth between these top and bottom surfaces is bounded by vertical planes, and has, by supposition, for its horizontal section a rectangle. The usual mode of computing the solidity in such a case, is to divide the whole mass into four triangular prisms, each of which has for its solidity the product of its horizontal section by one third the sum of its vertical edges.* Thus if the whole horizontal section be represented by $A$, and the several vertical edges by the letters attached to

Fig. 1.

* See the author's "Field-Book for Railroad Engineers," p. 93.

them in the figure, we shall have for the solidity of the four triangular prisms,

$$(1)\qquad \frac{A}{4}\times\frac{(h+h_1+c)+(h_1+h_2+c)+(h_2+h_3+c)+(h_3+h+c)}{3}.$$

This method is correct, but it compels us to interrupt the usual mode of calculating any number of adjacent prisms by one operation. The exceptional solid has to be calculated alone, and the labor of calculating the surrounding prisms is increased. It is proposed to show, that the exceptional solid may be calculated with the others, just as if no additional height had been necessary, and that a proper correction may be introduced afterwards.

In calculating a number of rectangular prisms by one operation, it is assumed that the solidity of each prism is equal to the product of its horizontal section by one fourth the sum of its vertical edges. Now we can so modify the expression (1) as to separate from it the product just mentioned, namely,

$$\frac{A}{4}(h+h_1+h_2+h_3).$$

For by collecting the like quantities in (1) it becomes

$$\frac{A}{4}\left[\tfrac{2}{3}(h+h_1+h_2+h_3)\right]+\frac{Ac}{3}$$

$$(2)\quad =\frac{A}{4}(h+h_1+h_2+h_3)+\frac{Ac}{3}-\frac{A}{4}\times\tfrac{1}{3}(h+h_1+h_2+h_3)$$

$$=\frac{A}{4}(h+h_1+h_2+h_3)+\frac{A}{3}\left(c-\frac{h+h_1+h_2+h_3}{4}\right).$$

Now, by including this solid in the general calculation with the adjacent prisms, we should take in the first term of (2), namely,

$$\frac{A}{4}(h+h_1+h_2+h_3),$$

leaving for the correction to be made the second term of (2), or

$$\frac{A}{3}\left(c-\frac{h+h_1+h_2+h_3}{4}\right).$$

This correction, it will be seen, represents a pyramid of base $A$, and altitude equal to the centre height minus the average of the four corner heights, and it is, therefore, very readily obtained. If the centre height were less than the average of the corner heights, the correction would be a minus quantity, as it should be.

Fig. 2.

An exceptional solid sometimes arises in another way. The surface of one of the rectangular divisions may differ considerably from a plane, but may admit of being divided into two planes meeting in one of the diagonals (fig. 2). This would give two triangular prisms, the solidities of which, with the former notation, would be

$$(3) \qquad \frac{A}{2} \times \frac{h + 2h_1 + h_2 + 2h_3}{3}.$$

To combine this solid with the adjacent prisms, we must again be able to separate from (3) the expression $\frac{A}{4}(h + h_1 + h_2 + h_3)$. This can be done, and (3) becomes

$$\frac{A}{4}(h + h_1 + h_2 + h_3) + \frac{A}{12}[(h_1 + h_3) - (h + h_2)].$$

So that we may include this solid also in the general calculation, if we make the correction

$$\frac{A}{12}[(h_1 + h_3) - (h + h_2)].$$

In this correction $h_1 + h_3$ represents the sum of the heights touched by the dividing diagonal, and $h + h_2$ represents the sum of the heights not touched by the diagonal, and the latter sum is always to be subtracted from the former. If, then, the sum of the heights touched by the diagonal is greater than the sum of the other two, the correction will be positive; but if the first sum is less than the second, the correction will be negative.

## DEMONSTRATION OF A THEOREM.

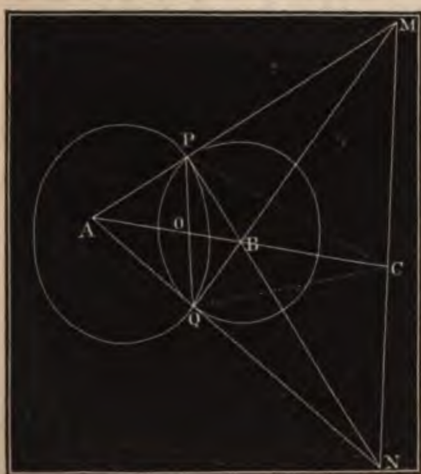
BY A. CAYLEY, F.R.S.
Associate Editor of the London Quarterly Journal of Pure and Applied Mathematics.

SIR, — The annexed demonstration of the very elegant geometrical theorem, Prize Problem No. V.* of the first number of the Mathematical Monthly, may perhaps be interesting to some of your readers, and if you should think it suitable, I shall be glad if you will insert it in your journal. I remain, sir, your obedient servant,

A. CAYLEY.

LONDON, 2 Stone Buildings, W. C., 23d Nov., 1858.

DEMONSTRATION. If, instead of the four circles, we consider any two of the circles, the triangle will be that formed by any three of the four common tangents of the two circles, and the theorem may be thus stated: "Given two circles, consider the triangle formed by any three of the four common tangents of the circles; the circle circumscribed about this triangle will pass through the middle point of the line of junction of the centres of the two circles." A circle may be considered as a conic intersecting the line infinity in two fixed points, the "circular points at infinity:" the centre of the circle is the pole of the line infinity with respect to the circle; the middle point of the line of junction of the two centres is the fourth harmonic with respect to the two centres and the point in which the line of junction meets the line infinity. Consider, then, any two conics intersecting in the points $P$, $Q$, and let the tangents at




* By H. A. NEWTON, Professor of Mathematics in Yale College, New Haven, Ct.

$P, Q$ to the first conic meet in the point $A$, and the tangents at $P, Q$ to the second conic meet in the point $B$, and let $P\,Q$ meet $A\,B$ in $O$, and $C$ be the fourth harmonic of $O$ with respect to the points $A$ and $B$: the point $C$ may also be determined as follows, namely, if the tangent at $P$ to the first conic meet the tangent at $Q$ to the second conic in $M$; and the tangent at $P$ to the second conic meet the tangent at $Q$ to the first conic in $N$; then $C$ is the point of intersection of $A\,B$ and $M\,N$. And we have the following theorem, namely: "Given two conics intersecting in the points $P$ and $Q$; let the point $C$ be determined as above; then considering the triangle formed by any three of the four common tangents of the two conics, the angles of this triangle, and the points $P, Q, C$ lie in one and the same conic;" a theorem which includes the theorem for two circles. And the analytical demonstration is very easy; for take $x = 0, y = 0, z = 0$ as the equations of the lines $A\,B, P\,Q$, and $M\,N$ respectively; the equations of the two conics may be written

$$x^2 - (y - z)^2 = 4\,\lambda^2 y^2,$$
$$x^2 - (y + z)^2 = 4\,\mu^2 y^2;$$

where $\lambda, \mu$ are arbitrary coefficients. The equations of the sides $P\,Q,\ Q\,C,\ C\,P$ of the triangle $P\,Q\,C$ would be $y = 0, z - x = 0, z + x = 0$ respectively. The equations of the common tangents of the two conics may be found in the ordinary manner, and if we put for shortness

$$A = \sqrt{(\lambda - \mu)^2 + 1},\ A_1 = \sqrt{(\lambda + \mu)^2 + 1},$$

then the equations of the four common tangents are found to be

$$A\ x + (\lambda + \mu)\,y + (\lambda - \mu)\,z = 0,$$
$$-A\ x + (\lambda + \mu)\,y + (\lambda - \mu)\,z = 0,$$
$$A_1 x + (\lambda - \mu)\,y + (\lambda + \mu)\,z = 0,$$
$$-A_1 x - (\lambda - \mu)\,y + (\lambda + \mu)\,z = 0.$$

The general equation of a conic passing through the angles of the triangle formed by three of these lines, say the first three, is

$$\frac{A}{\mathcal{A}x+(\lambda+\mu)y+(\lambda-\mu)z}+\frac{B}{-\mathcal{A}x+(\lambda+\mu)y+(\lambda-\mu)z}+\frac{C}{\mathcal{A}_1x+(\lambda-\mu)y+(\lambda+\mu)z}=0;$$

and the theorem will be true, if $A$, $B$, $C$ can be so determined that this conic may pass through the angles of the triangle

$$y=0,\quad z-x=0,\quad z+x=0.$$

If we substitute in the equation successively the values

$$y=0,\ z=x;\quad y=0,\ z=-x;\quad x=0,\ z=0;$$

we find

$$\frac{A}{\mathcal{A}+\lambda-\mu}+\frac{B}{-\mathcal{A}+\lambda-\mu}+\frac{C}{\mathcal{A}_1+\lambda+\mu}=0,$$

$$\frac{A}{\mathcal{A}-\lambda+\mu}+\frac{B}{-\mathcal{A}-\lambda+\mu}+\frac{C}{\mathcal{A}_1-\lambda-\mu}=0,$$

$$\frac{A}{\lambda+\mu}+\frac{B}{\lambda-\mu}+\frac{C}{\lambda+\mu}=0;$$

or attending to the values of $\mathcal{A}$ and $\mathcal{A}_1$, the equations are

$$A(\mathcal{A}-\lambda+\mu)+B(-\mathcal{A}-\lambda+\mu)+C(\mathcal{A}_1-\lambda-\mu)=0,$$

$$A(\mathcal{A}+\lambda-\mu)+B(-\mathcal{A}+\lambda-\mu)+C(\mathcal{A}_1+\lambda+\mu)=0,$$

$$A(\lambda-\mu)+B(\lambda-\mu)+C(\lambda+\mu)=0;$$

which are obviously equivalent to the two equations

$$A\mathcal{A}+B(-\mathcal{A})+C\mathcal{A}_1=0,$$

$$A(\lambda-\mu)+B(\lambda-\mu)+C(\lambda+\mu)=0,$$

and give

$$A:B:C=-\mathcal{A}(\lambda+\mu)-\mathcal{A}_1(\lambda-\mu):-\mathcal{A}(\lambda+\mu)+\mathcal{A}_1(\lambda-\mu):2\mathcal{A}(\lambda-\mu);$$

and the theorem is consequently proved.

## NOTES ON THE HYPOCYCLOID.

BY PROF. O. ROOT.

In the first Number of the Mathematical Monthly, Mr. Watson shows that the curve of equilibrium for a heavy rod, with one end against a smooth vertical wall, is the Hypocycloid; in which the radius of the generating circle is one fourth of that of the fixed circle; of which the equation is

$$(1) \qquad x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}},$$

$r$ being one half of the length of the rod. This elegant equation occurs in several problems of interest.

In Leyburn's Mathematical Repository, problem 318 is this: If from one of the angles of a rectangle, a perpendicular be drawn to its diagonal, and from their intersection, lines be drawn perpendicular to the sides containing the opposite angle, we shall have, by denoting these last perpendiculars by $x$ and $y$, and the diagonal by $r$, the equation (1).

In the Cambridge problems for 1803, it is required to find the longest ladder which can be slid up a vertical wall under an obstacle given in position. If we denote the coördinates of the obstacle by $(x\, y)$, and the length of the ladder by $r$, we shall get by the ordinary process, equation (1), which of course gives $r$ when $x$ and $y$ are known. In "Wright's Solutions of the Cambridge Problems," the solution of this problem is quite wrong.

A few years since, a problem was circulated among the teachers in this country, in which it was required to find the longest pole that can be run up a chimney, in which the height of the crossbar and the depth of the flue are given. This is obviously the same as the preceding.

In No. 4 of the Mathematical Miscellany, problem 7 is this: Required to find the locus of all the points so situated within a right angle, that the shortest line which can be made to pass through

each of them, and terminate in the sides of the right angle, shall be of a constant length. The equation of the required locus is readily found to be the same as (1).

This is also the equation of the locus of the consecutive intersections of a straight line of a given length, terminating in the sides of a right angle. If we take the origin at the right angle, then will

$$(2) \qquad y = ax + b$$

be the equation of the line of given length; whence we shall have, since for $y = 0$, $x = -\frac{b}{a}$, and for $x = 0$, $y = b$

$$b^2 + \frac{b^2}{a^2} = r^2,$$

in which $r$ is the given length.

With this eliminate $a$ from (2), then will

$$(3) \qquad y = \frac{b\,x}{(r^2 - b^2)^{\frac{1}{2}}} + b.$$

By differentiating (3) on the supposition that $b$ only is variable, we get

$$b = r\left(1 - \left(\frac{x}{r}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}},$$

and this value of $b$ substituted in (3) gives

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}}.$$

If, in the common equation of the Hypocycloid, we put $r' = \frac{1}{4}\,r$, we get

$$x = r \sin^3 \theta, \qquad y = r \cos^3 \theta;$$

and from these equations we obtain

$$\int (d\,x^2 + d\,y^2)^{\frac{1}{2}} = 3\,r \int \sin\theta \cos\theta \,d\,\theta = \tfrac{3}{2}\,r \sin^2\theta = \tfrac{3}{2}\,r,$$

for the length of the curve between the limits $\theta = 0$, and $\theta = 90°$. Also, for the area between the same limits we find

$$\int y\,d\,x = 3\,r^2 \int \sin^2\theta \cos^4\theta \,d\,\theta = \frac{3\,\pi\,r^2}{3\,2};$$

for the surface of the solid of revolution around the axis of $x$

$$\int 2\,\pi\,y\,d\,s = 6\,\pi\,r^2 \int \sin\theta \cos^4\theta \,d\,\theta = -\,\frac{6\,\pi\,r^2 \cos^5\theta}{5} = \frac{6\,\pi\,r^2}{5};$$

for the solidity of this solid of revolution,

$$\int \pi\,y^2\,d\,x = 3\,\pi\,r^3 \int \sin^2\theta \cos^7\theta \,d\,\theta = \frac{16\,\pi\,r^3}{105}.$$

## NOTE ON THE EQUATION OF THE PROJECTION OF A GREAT CIRCLE OF THE GLOBE ON A MERCATOR CHART.

BY PROF. W. CHAUVENET

LET the great circle make the angle $\alpha$ with the equator, and let its intersection with the equator be taken as the origin of rectangular coördinates. These coördinates on the chart are

$x =$ the longitude of a point of the projected curve, reckoned from the origin,

$y =$ the expanded arc of the meridian from the equator to the point.

If then $\varphi$ is the latitude of the point $(x, y)$, the radius of the sphere being unity, we have on the chart,

$$y = \text{Nap. log tan}\left(\frac{\pi}{4} + \tfrac{1}{2}\,\varphi\right),$$

and on the sphere,

$$\tan\varphi = \tan\alpha\,\sin x\,,$$

from which the equation of the projection is obtained as follows: If $e =$ the base of Naperian logarithms,

$$\tan\left(\frac{\pi}{4} + \tfrac{1}{2}\,\varphi\right) = e^{y}$$

$$\tan\left(\frac{\pi}{4} - \tfrac{1}{2}\,\varphi\right) = e^{-y},$$

the half difference of which is

$$\tan\varphi = \tfrac{1}{2}\,(e^{y} - e^{-y}) = \text{potential sine of } y,$$
$$= \text{Sin}\, y.$$

The required equation is, therefore,

$$\text{Sin}\, y = \tan\alpha\,\sin x.$$

The construction of the curve by points will then be easily effected with the aid of a table of potential functions.

## NOTES ON THE THEORY OF PROBABILITIES.

By Simon Newcomb. Naut. Alm. Office, Cambridge, Mass.

1. The theory of probabilities, or doctrine of chances, as it has sometimes been called, being one of those subjects on which little has yet been written among us, it is proposed to develop its elementary principles through the pages of this Journal, and to apply them to a number of such problems as are suggested either by the observation of the material world, or by the affairs of society and of daily life.

2. Against the logical accuracy of the results of this calculus many objections have been raised. In the latter part of the last century, the Rev. Mr. Mitchell computed, that if the stars visible to the naked eye, had been scattered fortuitously over the heavens, there were five hundred thousand chances to one against any six of them falling together in so small a space as that occupied by the Pleiades. The correctness of this conclusion has lately been disputed by an eminent English scientist, who thinks that in the case supposed, such groupings of the stars are precisely what ought to be expected. This problem will be examined in the course of these notes as a particular case of one more general, and we shall find by a mode of reasoning entirely different from that of Mr. Mitchell, that he is substantially correct.

3. The object of the theory of probabilities is to determine the amount of reason which we have to expect any event when we have not the data to determine with certainty whether it will occur or not; and when the data of which we are possessed are such as to admit of the application of mathematical analysis. This amount of reason is not any quality of the event itself, but only a degree of rational expectation in the mind of the individual.

It is considered a well-established philosophical principle that the same causes will always produce the same effects; so that an intellect having a perfect knowledge of the former, and possessed of sufficient reasoning power, could always predict the latter. But in the external world, we can never arrive at any knowledge of the philosophical efficient cause, but can only observe that certain events have always been followed by certain other events. The preceding event is, then, generally denominated the cause, and the event which follows it the effect. Using the term cause in this sense, it generally happens, that so much of the cause as we are cognizant of may be followed by a great number of effects; and then we can only determine of the latter that they are more or less probable.

4. Hereafter, the operation of so much of the cause as we are cognizant of will be called a *trial;* and the effects of which we wish to find the probabilities will be designated as *events.* Generally the number of separate possible results of the trial will be greater than the number of events which we consider. The former will then be designated as *cases;* such of the cases as necessarily involve or produce the event in question are called *favorable,* while those which necessarily negate or prevent it are termed *unfavorable.* If the separate cases are all equally probable, the probability of the event is represented by the ratio of the number of the favorable cases to the whole possible number of cases.

5. To illustrate the above principles and definitions: suppose that a cube has the sides numbered from one to six; that sides one and two are white, and the remaining four black. Suppose, also, that the cube is thrown from a box at random, and the probability that a white side will fall upward is required. Then —

The given cause or *trial* is the throwing of the cube.

The *event* is the falling upward of a white side.

The number of possible *cases* is six; namely, the falling up of sides 1, 2, 3, &c., to 6 respectively.

The *favorable* cases are two, the falling of side 1, and of side 2.

Wherefore the *probability* of the event is $\frac{2}{6} = \frac{1}{3}$.

6. In general, if there are $m$ cases equally probable, of which one must, and only one can, occur; and if $p$ of these cases are favorable to the production of an event, the probability of that event is $\frac{p}{m}$. We see that if the probability is $\frac{1}{2}$, the event is just as likely to occur as to fail; and that if the probability is 1, the event becomes certain. All degrees of probability must be included between the limits 0 and 1; 0 being the symbol of impossibility and 1 of certainty.

7. In enumerating the different possible cases, we must take care so to subdivide them that they shall all be equally probable. Through not taking this precaution, an eminent mathematician of the last century fell into an error with regard to a very simple problem, of the following nature: If a coin is thrown twice, what is the probability that a head will fall upward? D'ALEMBERT reasoned as follows: If head is thrown the first time, then the event occurs, and the throw need not be repeated; only three cases are therefore to be considered —

Heads the first time,
Tails the first time and heads the second,
Tails both times.

Two of these cases being favorable, the probability of the event is $\frac{2}{3}$. This result, however, is incorrect, because, as is easily seen, the probability of the first case alone is $\frac{1}{2}$, and therefore twice as great as that of either of the others. It ought, therefore, to be considered as subdivided into two separate cases, making the probability of the event $\frac{3}{4}$, which is the correct answer.

8. The following theorem will frequently save us the trouble of enumerating all the possible cases.

*The probability that both of two independent events will happen, is equal to the product of their individual probabilities.*

For the first event, let there be $m$ cases, of which $p$ are favorable, and for the second, $n$ cases, of which $q$ are favorable. If we give both events a trial, each of the $m$ cases may be combined with any one of the $n$ cases, making in all $m \times n$ possible combinations, each of which will be equally probable. The combinations favorable to *both* events will be those only in which one of the $p$ cases, favorable to the first, is combined with one of the $q$ cases, favorable to the second, the number of which will be $p \times q$. The probability of the compound event is therefore $\frac{pq}{mn}$, which is equal to the product of the separate probabilities.

As an example, suppose that with the cube spoken of in § 5 is put an octahedron, of which three sides only are white. If the two solids are thrown down at random, each of the six sides of the cube may be combined with any one of the eight sides of the octahedron, making the possible combinations (1, 1), (1, 2), (1, 3), &c. to (1, 8); (2, 1), (2, 2), (2, 3), &c. to (2, 8); .... (6, 1), (6, 2), &c. to (6, 8); in all, 48 combinations. If we determine the result of each of these combinations, we shall find that the upper side of

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| The | cube | will | be | white | and | the | octahedron | white | in | 6 | cases. |
| " | " | " | " | white | " | " | " | black | " | 10 | " |
| " | " | " | " | black | " | " | " | white | " | 12 | " |
| " | " | " | " | black | " | " | " | black | " | 20 | " |

Dividing each of these results by 48, the whole number of cases, we shall find the probabilities of the several compound events to be equal to the product of the probabilities of the simple events which compose them; that is, $\frac{6}{48} = \frac{2}{6} \times \frac{3}{8}$; $\frac{10}{48} = \frac{2}{6} \times \frac{5}{8}$; $\frac{12}{48} = \frac{4}{6} \times \frac{3}{8}$; $\frac{20}{48} = \frac{4}{6} \times \frac{5}{8}$.

THE

# MOTIONS OF FLUIDS AND SOLIDS

RELATIVE TO

# THE EARTH'S SURFACE.

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## INTRODUCTION.

Some of the results contained in the following pages were published about two years ago, in an essay in the Nashville Journal of Medicine and Surgery, edited by Professor W. K. Bowling, M. D., of Nashville, Tennessee. A small edition was also published in pamphlet form, and distributed by the Smithsonian Institution and myself amongst various scientific men, libraries, and scientific associations, both in this country and in Europe. In that essay it was attempted to show that the depression of the atmosphere at the poles and the equator, and the accumulation or bulging at the tropics, as indicated by barometric pressure, the gyratory motion of storms from right to left in the northern hemisphere, and the contrary way in the southern, and certain motions of oceanic currents, are necessary consequences of the modifying forces arising from the earth's rotation on its axis, and also that the observed flowing of the lower strata of the atmosphere in the middle latitudes towards the poles, contrary to the ordinary theory of the trade winds, is caused by the greater pressure of this accumulation of atmosphere at the tropics. It is believed that that essay was the first attempt to account for those remarkable phenomena by means of

the modifying influence of the earth's rotation, and that it furnishes the only satisfactory explanation of them which has yet been given.

In that essay it was inconvenient to use any mathematical formulæ, and consequently the results merely of only a partial and imperfect investigation of the subject were given; but it is thought that on account of the importance of the subject, it deserves a more thorough investigation. It is proposed, therefore, in the following pages, to go into a complete analytical investigation of the general motions of fluids surrounding the earth, and of projectiles at its surface, arising from disturbing forces and the earth's attraction, combined with the modifying forces arising from its rotation on its axis. We shall accordingly, in the first section, investigate the general equations of motion relative to the earth's surface, applicable to both fluids and solids, and in the subsequent sections treat, first of the motions and figure of the whole or a part of a fluid surrounding the earth, upon the hypothesis that its motions are not resisted by the earth's surface, and then apply the results thus obtained to the explanation of the general motions of the atmosphere, the motions of storms or hurricanes, and the currents of the ocean. We shall also give a complete but concise treatise on projectiles, taking into account the effect of the earth's rotation.

We hope to be able in this investigation to give a satisfactory explanation of all the general motions of the atmosphere and of the ocean; the cause of the greater pressure of the atmosphere near the tropics than at the equator and the poles, and of the greater pressure generally in the northern hemisphere than in the southern; to account for the motion of all great storms in both hemispheres from the equator towards the poles in parabolic paths, and to completely establish their gyratory character; none of which phenomena have ever been satisfactorily accounted for by any of the usual theories, which do not take into account the influence of the earth's rotation.

## SECTION I.

### OF THE GENERAL EQUATIONS OF MOTION RELATIVE TO THE EARTH'S SURFACE.

1. Let $x, y$, and $z$ be three rectangular coördinates, having their origin at the centre of the earth, $x$ corresponding with the axis of rotation. Also let

$\Omega$ be the potential of all the attractive forces of the earth,
$P$ the pressure of the fluid, and
$k$ its density.

Then $k\,D_x\,\Omega$, $k\,D_y\,\Omega$, and $k\,D_z\,\Omega$ are the forces for a unit of volume, arising from the earth's attraction, and $D_x\,P$, $D_y\,P$, and $D_z\,P$, those arising from the pressure of the fluid, in the directions respectively of $x, y$, and $z$; and we have for the equations of the absolute motions of the fluid, regarding the centre of the earth at rest,

$$
(1) \qquad \begin{aligned}
D_t^2\,x + D_x\Omega + \tfrac{1}{k}\,D_x\,P &= 0,\\
D_t^2\,y + D_y\Omega + \tfrac{1}{k}\,D_y\,P &= 0,\\
D_t^2\,z + D_z\Omega + \tfrac{1}{k}\,D_z\,P &= 0.
\end{aligned}
$$

Putting $P = 0$, they are the equations of a projectile.

2. Let $r$ be the distance from the earth's centre,
$\theta$, the polar distance,
$\varphi$, the longitude, and
$n$, the angular velocity of the earth's rotation.

Then we have

$$
(2) \qquad \begin{aligned}
x &= r\cos\theta,\\
y &= r\sin\theta\cos(nt + \varphi) = r\sin\theta\cos\omega,\\
z &= r\sin\theta\sin(nt + \varphi) = r\sin\theta\sin\omega,
\end{aligned}
$$

by putting for brevity $nt + \varphi = \omega$.

The position of the ordinates $y$ and $z$, and also the origin of the

time $t$, being entirely arbitrary, they must be so taken as to make $\sin(nt + \varphi)$ vanish in the plane of $x, y$.

Using these values of $x, y$, and $z$ in equations (1), we obtain equations in which the first derivatives of $r$, $\theta$, and $\varphi$ represent the motions of the fluid or projectile relative to the earth's surface.

3. Taking the first derivatives of (2) with regard to $t$, we get

$$D_t x = \cos\theta\, D_t r - r\sin\theta\, D_t\theta,$$
$$D_t y = \sin\theta\cos\omega\, D_t r + r\cos\theta\cos\omega\, D_t\theta - r\sin\theta\sin\omega\, D_t\omega,$$
$$D_t z = \sin\theta\sin\omega\, D_t r + r\cos\theta\sin\omega\, D_t\theta + r\sin\theta\cos\omega\, D_t\omega,$$

Taking the second derivatives, we get

$$D_t^2 x = \cos\theta\, D_t^2 r - 2\sin\theta\, D_t r\, D_t\theta - r\cos\theta\,(D_t\theta)^2 - r\sin\theta\, D_t^2\theta,$$

$$\begin{aligned} D_t^2 y = {} & \sin\theta\cos\omega\, D_t^2 r + 2\cos\theta\cos\omega\, D_t r\, D_t\theta - 2\sin\theta\sin\omega\, D_t\omega\, D_t r \\ (3)\quad & + r\cos\theta\cos\omega\, D_t^2\theta - r\sin\theta\cos\omega\,(D_t\theta)^2 - 2r\cos\theta\sin\omega\, D_t\omega\, D_t\theta \\ & - r\sin\theta\sin\omega\, D_t^2\varphi - r\sin\theta\cos\omega\,(D_t\omega)^2. \end{aligned}$$

$$\begin{aligned} D_t^2 z = {} & \sin\theta\sin\omega\, D_t^2 r + 2\cos\theta\sin\omega\, D_t r\, D_t\theta + 2\sin\theta\cos\omega\, D_t\omega\, D_t r \\ & + r\cos\theta\sin\omega\, D_t^2\theta - r\sin\theta\sin\omega\,(D_t\theta)^2 + 2r\cos\theta\cos\omega\, D_t\omega\, D_t\theta \\ & + r\sin\theta\cos\omega\, D_t^2\varphi - r\sin\theta\sin\omega\,(D_t\omega)^2. \end{aligned}$$

Since $x, y$, and $z$ are functions of $r$, $\theta$, and $\varphi$, we must put

$$\begin{aligned} & D_x\Omega = D_r\Omega\,.\,D_x r + D_\theta\Omega\,.\,D_x\theta + D_\varphi\Omega\,.\,D_x\varphi, \\ (4)\qquad & D_y\Omega = D_r\Omega\,.\,D_y r + D_\theta\Omega\,.\,D_y\theta + D_\varphi\Omega\,.\,D_y\varphi, \\ & D_z\Omega = D_r\Omega\,.\,D_z r + D_\theta\Omega\,.\,D_z\theta + D_\varphi\Omega\,.\,D_z\varphi. \end{aligned}$$

Now we have

$$r^2 = x^2 + y^2 + z^2,$$
$$\tan\theta = \frac{\sqrt{y^2+z^2}}{x},$$
$$\tan\omega = \frac{z}{y}.$$

Hence,

$$D_x r = \frac{x}{r} = \cos\theta,$$
$$D_y r = \frac{y}{r} = \sin\theta\cos\omega,$$
$$D_z r = \frac{z}{r} = \sin\theta\sin\omega,$$

$$D_x\theta = -\frac{\sqrt{y^2+z^2}}{r^2} = -\frac{\sin\theta}{r},$$

$$D_y\theta = \frac{xy}{r^2\sqrt{y^2+z^2}} = \frac{\cos\theta\cos\omega}{r},$$

$$D_z\theta = \frac{xz}{r^2\sqrt{y^2+z^2}} = \frac{\cos\theta\sin\omega}{r},$$

$$D_x\varphi = 0,$$

$$D_y\varphi = \frac{-z}{y^2+z^2} = \frac{\sin\omega}{r\sin\theta},$$

$$D_z\varphi = \frac{y}{y^2+z^2} = \frac{\cos\omega}{r\sin\theta}.$$

By means of these equations, equations (4) become

$$D_x\Omega = D_r\Omega\cos\theta - D_\theta\Omega\frac{\sin\theta}{r},$$

$$(5)\quad D_y\Omega = D_r\Omega\sin\theta\cos\omega + D_\theta\Omega\frac{\cos\theta\cos\omega}{r} - D_\phi\Omega\frac{\sin\omega}{r\sin\theta}.$$

$$D_z\Omega = D_r\Omega\sin\theta\sin\omega + D_\theta\Omega\frac{\cos\theta\sin\omega}{r} + D_\phi\Omega\frac{\cos\omega}{r\sin\theta}.$$

In the same manner we obtain

$$D_x P = D_r P\cos\theta - D_\theta P\frac{\sin\theta}{r},$$

$$(6)\quad D_y P = D_r P\sin\theta\cos\omega + D_\theta P\frac{\cos\theta\cos\omega}{r} - D_\phi P\frac{\sin\omega}{r\sin\theta},$$

$$D_z P = D_r P\sin\theta\sin\omega + D_\theta P\frac{\cos\theta\sin\omega}{r} + D_\phi P\frac{\cos\omega}{r\sin\theta}.$$

Substituting the values of the first members of equations (3), (5), and (6) in equations (1), and multiplying them respectively by $\cos\theta$, $\sin\theta\cos\omega$, $\sin\theta\sin\omega$, and adding, we obtain the first of the following equations. Again, multiplying them respectively by $r\sin\theta$,—$r\cos\theta\cos\omega$, and $-r\cos\theta\sin\omega$, and adding, we obtain the second of those equations. Finally, multiplying the last two respectively by $r\sin\theta\sin\omega$, and $-r\sin\theta\cos\omega$, and adding, we get the last of the following equations:—

$$\frac{1}{k}D_r P = -D_t^2 r + r(D_t\theta)^2 + r\sin^2\theta(n + D_t\omega)D_t\varphi$$
$$+ rn^2\sin^2\theta - D_r\Omega,$$

$$(7)\quad \frac{1}{k} D_\theta P = - r^2 D_t^2 \theta - 2 r D_t r \, D_t \theta + r^2 \sin\theta \cos\theta \,(n + D_t \omega)\, D_t \varphi$$
$$+ r^2 n^2 \sin\theta \cos\theta - D_\theta \Omega,$$
$$\frac{1}{k} D_\phi P = - r^2 \sin^2\theta \, D_t^2 \varphi - 2 r \sin^2\theta \, D_t \omega \, D_t r$$
$$- 2 r^2 \sin\theta \cos\theta \, D_t \omega \, D_t \theta - D_\phi \Omega.$$

In these equations $D_r P$, $\frac{1}{r} D_\theta P$, and $\frac{1}{r \sin\theta} D_\phi P$ represent the forces arising from the pressure in the directions respectively of $r$, $\theta$, and $\varphi$.

4. If $N$ be the normal distance to the surface of the earth, or to any level surface, and the forces in the first two of the preceding equations be resolved in the directions of the normal and a perpendicular to the normal in the plane of the meridian, putting $\cos \frac{r}{N} = 1$, and neglecting the small terms multiplied by $\sin \frac{r}{N}$, which are of the second order of the earth's ellipticity, and letting $\frac{1}{r} D_{\theta'} P$ represent the force arising from the pressure, resolved in the direction of the perpendicular to the normal in the plane of the meridian, the preceding equations give, when the fluid is at rest,

$$(8)\quad \begin{aligned} \frac{1}{k} D_N P &= r n^2 \sin^2\theta - D_N \Omega = - g, \\ \frac{1}{k} D_{\theta'} P &= r^2 n^2 \sin\theta \cos\theta - D_{\theta'} \Omega = 0, \\ \frac{1}{k} D_\phi P &= - D_\phi \Omega = 0, \end{aligned}$$

and hence, neglecting the very small terms multiplied by $\sin \frac{r}{N}$, depending upon the motions of the fluid relative to the earth's surface, they give for the fluid in motion,

$$\frac{1}{k} D_N P = - D_t^2 r + r (D_t \theta)^2 + r \sin^2\theta \,(n + D_t \omega)\, D_t \varphi - g,$$
$$(9)\quad \frac{1}{k} D_{\theta'} P = - r^2 D_t^2 \theta - 2 r D_t r \, D_t \theta + r^2 \sin\theta \cos\theta \,(n + D_t \omega)\, D_t \varphi,$$
$$\frac{1}{k} D_\phi P = - r^2 \sin^2\theta \, D_t^2 \varphi - 2 r \sin^2\theta \, D_t \omega \, D_t r$$
$$- 2 r^2 \sin\theta \cos\theta \, D_t \omega \, D_t \theta.$$

Integrating, we obtain

$$P = H - \int_N g\,k = H - g\,k\,N + \int_N N\,D_N(g\,k),$$
$$(10) \qquad = H - g\,k\,N + \int_N g\,N\,D_N k + \int_N N\,k\,D_N g,$$
$$= H + K,$$

in which $H$ must satisfy the following equations of partial differentials:—

$$\frac{1}{k} D_N H = -D_t^2 r + r\,(D_t\,\theta)^2 + r\,\sin^2\theta\,(n + D_t\,\omega)\,D_t\,\varphi,$$
$$(11)\ \frac{1}{k} D_{\theta'} H = -r^2 D_t^2\,\theta - 2\,r\,D_t r\,D_t\theta + r^2 \sin\theta\,\cos\theta\,(n + D_t\,\omega)\,D_t\,\varphi,$$
$$\frac{1}{k} D_\varphi\,H = -r^2 \sin^2\theta\,D_t^2\,\varphi - 2\,r\,\sin^2\theta\,D_t\,\omega\,D_t\,r$$
$$- 2\,r^2 \sin\theta\,\cos\theta\,D_t\,\omega\,D_t\,\theta,$$

and in which

$$(12) \qquad K = -g\,k\,N + \int_N g\,N\,D_N k + \int_N N\,k\,D_N g.$$

Hence $H$ is the pressure arising from the motions of the fluid, and $K$ that arising from its gravity. If $g$ and $k$ are functions of $N$, $\theta'$, and $\varphi$, $K$ is a function of the same.

5. For a stratum of equal pressure, $P$ is constant, and hence

$$D_{\theta'} P = 0, \qquad D_\varphi\,P = 0.$$

If we therefore put $K'$ and $h$ for the special values respectively of $K$ and $N$, belonging to a stratum of equal pressure, and take the derivatives of (10) with regard to $\theta'$ and $\varphi$, and neglect the very small terms containing $D_t\,r$ as a factor, which, in all ordinary motions of the fluid, will be shown to be insensible, we obtain for the general equations of horizontal motions, by restoring the value of $\omega$ in § (2).

$$0 = \frac{1}{k} D_{\theta'}\,K' - r^2\,D_t^2\,\theta + r^2 \sin\theta\,\cos\theta\,(2\,n + D_t\,\varphi)\,D_t\,\varphi,$$
$$(13)$$
$$0 = \frac{1}{k} D_\varphi\,K' - r^2 \sin^2\theta\,D_t^2\,\varphi - 2\,r^2 \sin\theta\,\cos\theta\,(n + D_t\,\varphi)\,D_t\,\theta,$$

in which

$$(14) \qquad K' = -g\,k\,h + \int_h g\,h\,D_h k + \int_h h\,k\,D_h g.$$

6. If we suppose the fluid to be elastic, and the ratio of the density to the elastic force or pressure to depend upon the temperature, we may put

(15) $$k = \alpha P,$$

in which $\alpha$ may be a function of $h$, $\theta'$, and $\varphi$. Substituting this value of $k$ in (14), we get, when $g$ may be regarded as constant, since in that case the last term of (14) vanishes.

(16) $$\begin{aligned} D_{\theta'} K' &= - D_{\theta'} (g \alpha P h) + D_{\theta'} \textstyle\int_h g h D_h (\alpha P), \\ &= - g \alpha P D_{\theta'} h - g h P D_{\theta'} \alpha + g P D_{\theta'} \textstyle\int_h h D_h \alpha, \end{aligned}$$

Hence, dividing by $k = \alpha P$, we get

(17) $$\begin{aligned} \tfrac{1}{k} D_{\theta'} K' &= - g D_{\theta'} h - g h D_{\theta'} \log \alpha + A_{\theta'}, \\ \tfrac{1}{k} D_{\varphi} K' &= - g D_{\varphi} h - g h D_{\varphi} \log \alpha + A_{\varphi}, \end{aligned}$$

by changing $\theta'$ to $\varphi$ for the last equation, and putting

(18) $$\begin{aligned} A_{\theta'} &= \tfrac{g}{\alpha} D_{\theta'} \textstyle\int_h h D_h \alpha, \\ A_{\varphi} &= \tfrac{g}{\alpha} D_{\varphi} \textstyle\int_h h D_h \alpha. \end{aligned}$$

7. When $D_h \alpha$ is constant, that is, when $\alpha$ varies as the altitude,

(19) $$\begin{aligned} A_{\theta'} &= \tfrac{g e}{2 \alpha} D_{\theta'} h^2, \\ A_{\varphi} &= \tfrac{g e}{2 \alpha} D_{\varphi} h^2, \end{aligned}$$

in which

$$e = D_h \alpha,$$

depending upon the constant ratio of increase or decrease of $\alpha$ with $h$.

8. When $\alpha$ is constant, or when the fluid is homogeneous, the last two terms of (17) vanish, and in the latter case, $h$ may represent the height of the surface of the fluid above any level surface.

9. In the preceding investigation, the effect upon $g$ arising from a change of the figure of the fluid has been neglected. It is small

in the case of water, and in the case of the atmosphere, entirely insensible.

10. Equations (13), together with the condition that the same amount of fluid must always occupy a space which is inversely as its density, the analytical expression of which is called the equation of continuity, are the conditions which must be satisfied by the motions of a fluid surrounding the earth, and are sufficient to determine its horizontal motions, and also the value of $h$, which gives the figure of the fluid. When $h$ is determined, equation (10) gives the pressure of the fluid. As only a very special form of the general equation of continuity will be needed in this investigation, it is unnecessary to give it here.

[TO BE CONTINUED.]

## PROBLEM.

BY J. FOSTER FLAGG, C. E.

REQUIRED a formula for the strength of a circular flat iron plate of uniform thickness, supported throughout its circumference and uniformly loaded; likewise a formula, supposing the plate to be bolted down. Also find the equation of the curve, for each case, which should be given to a section of the plate, in order that it may have the greatest strength with the least material. The problem may be further varied by supposing the plate *square*, instead of round.

We have recently had great need of such formulas on the Washington Aqueduct, and if they were obtained in a moderately simple shape, they would, I think, be quite valuable. I know of no place where the discussion of these questions may be found, although it is quite possible that it may have been made.

# Mathematical Monthly Notices.

I. *Examples of the Processes of the Differential and Integral Calculus.* Collected by D. F. Gregory, M. A., late Fellow of Trinity College. Second Edition, edited by William Walton, M. A., Trinity College.

II. *Problems in Illustration of the Principles of Plane Coördinate Geometry.*

III. *A Collection of Problems in Illustration of the Principles of Theoretical Mechanics.* Second Edition, with numerous Alterations and Additions.

IV. *A Collection of Problems in Illustration of the Principles of Theoretical Hydrostatics and Hydrodynamics.*

V. *A Collection of Problems in Illustration of the Principles of Elementary Mechanics.* By William Walton, M. A., Trinity College, Cambridge; Mathematical Lecturer at Magdalene College. Cambridge: Deighton, Bell & Co.

We include these works prepared by Mr. Walton in the same notice, because their aim and spirit are the same; and any remarks applicable to one will apply with more or less force to all. The simple titles give a good idea of the place they are intended to fill, and it only remains to call attention to the plan upon which they are constructed.

In the Preface to II., Mr. Walton says: "The tendency of even detached and desultory problems is no doubt to sharpen the intellect of the learner, and to give him a true conception of the signification of corresponding and more abstract propositions; the classification, however, of problems by some law of affinity is, in my opinion, calculated not only to produce the same result in a still higher degree, but also to qualify him by an enlarged comparison for becoming skilful and judicious in effecting his own solutions. In this conviction has originated the composition of the present work."

This quotation clearly shows the one governing principle which the author has kept steadily in view in their preparation. We have long felt, that, especially for the student who wishes such a thorough knowledge of the higher mathematics as will enable him to use them successfully as an instrument, something more was needed than the simple development of theory, with applications to a few special problems intended to elucidate it, rather than show its power as a means of investigation; that collections of problems were needed in the various branches of pure and applied mathematics, so graded with respect to difficulty as always to tax his ability yet never to discourage his efforts, and so classified with regard to the principles involved, that he may always be sure of his knowledge, though he may often want the skill to solve them; that such aid should be given in the solution of problems of special difficulty, with such general suggestions upon all, as would stimulate effort and not preclude its necessity; that such problems should be especially noticed as have an historical value, and the solutions of which have been instrumental in inventing or perfecting theory; that such references should be made to the original sources of information respecting the authorship and full discussion of typical or characteristic problems, as would enable the student to consult them for himself. In the above volumes these wants are, to a good degree, supplied.

For instance, Volume II. is arranged under the following heads; each head being divided into sections which are severally devoted to a special property. Straight Line, in 7 sections, including 143 problems; Circle, in 12 sections, with 93 problems; Parabola, in

20 sections, with 128 problems; ELLIPSE, in 27 sections, with 202 problems; HYPERBOLA, in 20 sections, with 135 problems; LINES OF THE SECOND ORDER, in 28 sections, with 171 problems.

Besides this large number of interesting problems, there are many elegant solutions, with over five hundred references to all the Mathematical Journals, as well as authors, from Apollonius down to the present time. A like analysis would show the other volumes to be, in all important particulars, equally valuable. They are all intended as supplementary to any of the text-books upon the respective subjects, and to be used in connection with them.

The science of mathematics, and the *science of teaching* mathematics, are very different; and while Mr. WALTON may not claim to have added much to the former science by preparing these volumes, he may fairly claim that he has at least fully developed an important and fundamental principle in any thorough system of mathematical instruction, and contributed, in no small degree, towards elevating the profession of teaching to the dignity of a science. Instead of simply studying mathematics as an end to themselves, or as the means to a more material end, he has also studied them in their relations to the laws which govern mental growth, and as a means of mental culture and development. If this brief notice shall bring these volumes to the attention of that class of students who can appreciate and profit by them, our object in noticing them will be accomplished.

*The Lady's and Gentlemen's Diary; or, Poetical and Mathematical Almanac for the year* 1859. London: Printed for the Company of Stationers.

This is the 156th number of this celebrated Annual. Besides the solutions of the fifteen problems proposed in the number for 1858, we find fifteen new ones, the solutions of which will be published in the next number. We quote No. XV., hoping that some of our readers may be induced to compete for the prize offered for its solution.

PRIZE QUESTION; by Mr. C. H. BROOKS, C. E., Newcastle-upon-Tyne.

"Two rods, two and four feet long respectively, having their middle points connected by a string one foot in length, are thrown up; show that the chance of their crossing is $\frac{1}{2} + \frac{2}{\pi^2}$."

The number closes with interesting "Mathematical Papers" by STEPHEN FENWICK, F. R. A. S., of the Royal Military Academy, Woolwich; T. T. WILKINSON, F. R. A. S.; W. S. B. WOOLHOUSE, London; and Mr. SAMUEL BILLS, of Hawton, near Newark-upon-Trent. To some of these papers we shall have occasion to refer more particularly hereafter.

## Editorial Items.

*Solutions of the Prize Problems in the October Number of the Monthly.*

THE solutions which we have the pleasure of announcing were, much to our regret, placed in the hands of the Committee at too late a date to allow sufficient time to report in this number of the Monthly. The report of the Committee, with the Prize Solutions, will be published in the next number. We cannot, however, omit this opportunity to express our satisfaction, with the unmistakable evidence before us, of the good influence our prizes are exerting; and we feel sure, that no one who has sent us solutions will regret the labor he has expended upon them, whether successful in securing a prize or not. It must not be forgotten, however,

that the best solution of each problem is published as the "prize solution," thus making it desirable for students to send even a single solution. We are much obliged to those who have sent us solutions, marked "not for a prize;" and shall be glad if all taking sufficient interest in the Prize Problems to solve them, will send us their solutions. Lastly, we beg those whose names are found in the following announcement to accept our sincere thanks for the interest they have taken in this department of the Monthly.

Cleveland Abby, Student in Astronomy, University of Michigan, answered all the questions. (Dr. Brünnow, Prof.)

George A. Osborne, Jr., Student in the Lawrence Scientific School, answered all the questions. (H. L. Eustis, Prof.)

G. W. Hill, Class of 1859, Rutger's College, New Brunswick, N. J., answered all the questions. (Dr. Strong, Prof.)

Student in Brown University, Providence, R. I., answered all the questions. (Name of student not received.)

Arthur Wilkinson, Junior Class, Harvard College, answered all but Prob. V. (Benjamin Peirce, Prof.)

Joseph Turnbull, Salineville, Columbiana Co., answered all but Prob. II.

E. P. Austin, Student in the University of Michigan, answered all the questions. (John E. Clark, Prof.)

Asher B. Evans, Junior Class, Madison University, Hamilton, N. Y., answered all the questions. (L. M. Osborn, Prof.)

H. G. Palfrey, Junior Class, Harvard College, answered all the questions. (Benjamin Peirce, Prof.)

Charles W. Hassler, Student in Columbian College, Washington, D. C., answered all the questions. (Edward T. Fristoe, Prof.)

George W. Fisher, Senior Class, Yale College, New Haven, Ct., answered all the questions. (H. A. Newton, Prof.)

Francis E. Tower, Junior Class, Amherst College, Amherst, Mass., answered all the questions. (E. S. Snell, Prof.)

H. F. Fisk, Junior Class, Wesleyan University, Middletown, Ct., answered all but Prob. III. (J. M. Van Vleck, Prof.)

George B. Hicks, Student, Cleveland, Ohio, answered all the questions.

J. A. Williamson, Student in Normal College, North Carolina, answered all the questions. (Lemuel Johnson, Prof.)

William Everett, Senior Class, Harvard College, answered all the questions. (Benjamin Peirce, Prof.)

William C. Cleveland, Student in the Lawrence Scientific School, answered all the questions. (H. L. Eustis, Prof.)

Waller Holladay, Student of Mathematics in the University of Va., answered all the questions. (A. T. Bledsoe, Prof.)

Theodore Cooper, Cooper's Plains, New York, answered all but Prob. II.

W. F. Osborne, Sophomore Class, Wesleyan University, answered all the questions. (J. M. Van Vleck, Prof.)

Joseph Metzger, Student in Commercial and Classical Institute, New York, answered Probs. II. III. IV. (John Livor, Principal.)

P. Barton, Student, Amsterdam, New York, answered all but Prob. V.

W. R. McCrary, Student in the Military Institute, Franklin Co., Ky., answered all the questions. (R. J. Adcock, Prof.)

J. GIBSON CANAN, Student in Washington College, Chestertown, Md., answered all but Prob. V., and the first equation in Prob. I.

WILLIAM P. TEN BROECK, Student, La Fayette, Ind., answered Probs. I. and II.

GEORGE W. PIERCE, Student in the Classical and Mathematical School of Profs. Lovering and Lane, Cambridge, Mass., answered all the questions, but equation 3, Prob. I.

JOSEPH P. FRIZELL, Student in the office of J. B. Henck, Civil Engineer, Boston, answered all the questions but equation 3, Prob. I.

O. B. WHEELER, Student in the University of Michigan, Ann Arbor, answered all the questions. (D. Wood, Prof.)

F. W. BARDWELL, B. S., a graduate of the Lawrence Scientific School, Cambridge, Mass., and Assistant in the office of the American Ephemeris and Nautical Almanac, has accepted the appointment of Professor of Mathematics, Astronomy, and Civil Engineering in Antioch College, Yellow Springs, Ohio, and will enter upon the duties of the office Jan. 5, 1859. We are satisfied that the College has made a wise selection, and that Mr. BARDWELL will prove himself a thorough and faithful as well as able teacher. . . . . . . Authors will much oblige us if they will draw their figures just as they wish them engraved, and upon as small a scale as is consistent with clearness. For size, see cuts in the Monthly. . . . . . . All problems, especially those intended for insertion among the Prize Problems, should be accompanied with their solutions, to enable us readily to judge of their fitness for the purpose; otherwise their insertion must generally be delayed until we have time to examine them. We hope our friends will continue to send us such original problems as may from time to time occur to them. . . . . . . The desirableness of giving each author's profession and location was not suggested to us until some of the pages of our present number were electrotyped; which will account for the want of uniformity in this respect. . . . . . . In the demonstration of Prop. 7, page 84, there is a slight oversight. The constant $C$ equals zero, instead of $\frac{1}{6}a^3$. The removal of this term from the equation will give the correct values of $x = \frac{(1 \pm \frac{1}{2}\sqrt{2})a}{3}$ . . . . . . . It gives us pleasure to add the following names to our list of coöperators and contributors: JOHN MASON BROWN, Esq., Frankfort, Ky.; GEORGE W. HILAND, Esq., Adams, Seneca Co., Ohio; Prof. O. H. LELAND, Baylor University, Independence, Texas; ARTEMAS MARTIN, Esq., Franklin, Venango Co., Penn.; GEORGE STUART, Esq., Tutor in Haverford College, West Haverford, Penn.; Captain D. P. WOODBURY, U. S. Corps of Engineers, Tortugas, Florida. . . . . . . BOOKS RECEIVED. — Papers on Practical Engineering, published by the Engineer Department, for the Use of the Officers of the United States Corps of Engineers. — No. 1. Bitumen; by Lieut. H. WAGER HALLECK, U. S. Corps of Engineers. (Out of Print). — No. 2. Sea-Wall, on Lovell's Island; by Brevet-Colonel S. THAYER, U. S. Corps of Engineers. Washington, 1844. — No. 3. Sustaining Walls; by Lieut. D. P. WOODBURY, U. S. Corps of Engineers. Washington: Taylor & Maury, 1854. — No. 4. Military Bridges; by Capt. GEORGE W. CULLUM, U. S. Corps of Engineers. New York: D. Appleton & Co., 1849. — No. 5. On the Resistance of Piles to Superincumbent Pressure; by Brevet Lieut.-Colonel JAMES L. MASON, Capt. U. S. Engineers. Washington, 1850. — No. 6. On the Effects or Firing with Heavy Ordnance from Casemate Embrasures, and also the Effects of Firing against the same Embrasures with various Kinds of Missiles; by Brevet Brigadier-General JOSEPH G. TOTTEN, Colonel and Chief Engineer U. S. Army. Washington: Taylor & Maury, 1857. — No. 7. Treatise on the various Elements of Stability in the Well-Proportioned Arch, with numerous Tables of the Ultimate and Actual Thrust; by Capt. D. P. WOODBURY, U. S. Corps of Engineers. New York: D. Van Nostrand, 1858. — Nouvelles Annales de Mathématiques, Novembre, 1858.

THE

# MATHEMATICAL MONTHLY.

Vol. I....FEBRUARY, 1859....No. V.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

THE abscissa and double ordinate of a segment of a common parabola are $a$ and $b$, and the diameters of its circumscribed and inscribed circles $D$ and $d$; to prove that $D + d = a + b$.

### II.

A great circle of the sphere passes through two given points; find the rectangular coördinates of its pole.

### III.

If the two sides of a movable right angle are always tangents to a given ellipse, its summit will describe a circle concentric with the ellipse, the radius of which is equal to the chord joining the extremities of the major and minor axes.

### IV.

If a circle be described through the foci of an ellipse and any point in the conjugate axis produced; to prove that the right line joining that point and one of the points where the circle cuts the ellipse, will be a tangent to the ellipse.

V.

If $D$ represent any diameter of an ellipse, and $P$ the parameter of $D$, to find when $D+P$ is the least, and when the greatest, possible.

The solution of these problems must be received by the first of April, 1859.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS, IN NO. I. VOL. I.

THE first Prize is awarded to G. W. HILL, of the Senior Class of Rutgers' College, New Brunswick, N. J.

The second Prize is awarded to WALLER HOLLADAY, Student of Mathematics, in the University of Virginia.

*Prize Solution of Problem I.*

"Find $\theta$ from each of the equations

(1) $$\tan\theta\tan 2\theta+\cot\theta=-2,$$

(2) $$2\sin^2 3\theta+\sin^2 6\theta=2,$$

(3) $$\cos n\theta+\cos(n-1)\theta=\cos\theta."$$

Equation (1) multiplied by $\tan\theta$ becomes

(4) $$\tan^2\theta\tan 2\theta+1=-2\tan\theta;$$

and since $$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta},$$

whence $$2\tan\theta=\tan 2\theta-\tan^2\theta\tan 2\theta,$$

we obtain, by substituting this value of $2\tan\theta$ in (4), and reducing

$$\tan 2\theta=-1$$

$$\therefore 2\theta=\tan^{-1}(-1)=n\pi+135^\circ=n\pi+\tfrac{3}{4}\pi\ \therefore\theta=\frac{(4n+3)\pi}{8}.$$

All the roots are obtained by giving $n$ all positive integer values.

Equation (2), since

$$2\sin^2 3\theta=1-\cos 6\theta,\text{ and }\sin^2 6\theta=1-\cos^2 6\theta,$$

becomes $$\cos^2 6\theta + \cos 6\theta = 0,$$
from which we get $\cos 6\theta = 0$, and $\cos 6\theta = -1$.

$$\therefore\ 6\theta = \cos^{-1} 0 = n\pi + 90° = n\pi + \tfrac{1}{2}\pi\ \therefore\ \theta = \frac{(2n+1)\pi}{12}$$

$$\therefore\ 6\theta = \cos^{-1}(-1) = 2n\pi + 180° = 2n\pi + \pi\ \therefore\ \theta = \frac{(2n+1)\pi}{6}.$$

Equation (3), since

$$\cos n\theta = \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta$$

$$\cos(n-2)\theta = \cos(n-1)\theta\cos\theta + \sin(n-1)\theta\sin\theta,$$

becomes by adding

$$\cos n\theta + \cos(n-1)\theta = 2\cos(n-1)\theta\cos\theta = \cos\theta;$$

therefore $\cos\theta = 0$, or $\cos(n-1)\theta = \tfrac{1}{2}$.

$$\therefore\ \theta = \cos^{-1} 0 = n\pi + 90° = \frac{(2n+1)\pi}{2},$$

$$\therefore\ (n-1)\theta = \cos^{-1}\tfrac{1}{2} = 2m\pi \pm 60° = 2m\pi \pm \tfrac{1}{3}\pi\ \therefore\ \theta = \frac{(6m\pm 1)\pi}{3(n-1)}.$$

These solutions are by Mr. O. B. Wheeler, student in the University of Michigan.

*Prize Solution of Problem II.*

"The whole surface of a right cone is three times the area of the base. Find the vertical angle."

The convex surface is twice the area of the base. But the convex surface = perimeter of base × ½ slant height, and the area of base = perimeter × ½ radius. Therefore, since one area is twice the other, slant side = twice radius of base. ∴ the section containing the axis is an equilateral triangle, and the vertical angle is 60°.

This solution was given by Messrs. Everett, Hill, Palfrey, and Tower.

*Prize Solution of Problem III.*

"The sum of the squares of the reciprocals of two radii vectores from the centre of an ellipse at right angles to each other is constant; the perpendicular from the centre, on the chord joining their extremities, is also constant. What part of the area of the ellipse is the circle whose radius is this perpendicular?"

The equation of the ellipse, referred to its centre and axes, is $A^2 y^2 + B^2 x^2 = A^2 B^2$, which, for $x = r \cos\varphi$ and $y = r \sin\varphi$, becomes its polar equation,

$$r^2 = \frac{A^2 B^2}{A^2 \sin^2\varphi + B^2 \cos^2\varphi};$$

the centre being the pole, and the transverse axis the prime radius. For $r'$, the radius vector perpendicular to $r$, $\varphi$ must be increased by $\frac{1}{2}\pi$.

$$\therefore\ r'^2 = \frac{A^2 B^2}{A^2 \sin^2(\varphi + \frac{1}{2}\pi) + B^2 \cos^2(\varphi + \frac{1}{2}\pi)} = \frac{A^2 B^2}{A^2 \cos^2\varphi + B^2 \sin^2\varphi},$$

$$\therefore\ \frac{1}{r^2} + \frac{1}{r'^2} = \frac{A^2 + B^2}{A^2 B^2} = \frac{r'^2 + r^2}{r'^2 r^2} = \text{a constant.}$$

Let $p$ denote the perpendicular; $\sqrt{r^2 + r'^2}$ is the chord; therefore

$$p\sqrt{r^2 + r'^2} = r r',$$

since both expressions denote the double area of the same triangle.

$$\therefore\ p^2 = \frac{r^2 r'^2}{r^2 + r'^2} = \frac{A^2 B^2}{A^2 + B^2} = \text{a constant.}$$

But $A B \pi =$ area of ellipse, and $\frac{A^2 B^2 \pi}{A^2 + B^2} =$ area of circle.

$$\therefore\ \frac{\text{area of circle}}{\text{area of ellipse}} = \frac{A B}{A^2 + B^2} = \text{the required part.}$$

If from the extremity $B$ of the conjugate axis lines be drawn to $A$ and $A'$ the extremities of the transverse axis, and the angle $A B A' = \theta$, then

$$\frac{A}{\sqrt{A^2 + B^2}} = \sin\tfrac{1}{2}\theta, \text{ and } \frac{B}{\sqrt{A^2 + B^2}} = \cos\tfrac{1}{2}\theta.$$

$$\therefore\ \frac{A B}{A^2 + B^2} = \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta = \tfrac{1}{2}\sin\theta.$$

$$\therefore\ \text{area of circle} = \tfrac{1}{2}\sin\theta \times \text{area of ellipse.}$$

This solution combines those given by Messrs. EVANS and EVERETT; the former gave the method of finding $p^2$, and the latter showed that the ratio of the areas equals $\frac{1}{2}\sin\theta$.

*Prize Solution of Problem IV.*

"Two circles, whose radii are $R$ and $r$, touch each other externally. If $\theta$ is the angle included between the common tangents to the two circles, prove that $\sin\theta = \frac{4(R-r)\sqrt{Rr}}{(R+r)^2}$."

The construction readily shows, that in the right triangle of which the hypothenuse and a side are $R + r$ and $R - r$, that the angle opposite $R - r$ is $\frac{1}{2}\theta$. Therefore

$$\sin\tfrac{1}{2}\theta = \frac{R-r}{R+r}, \text{ and } \cos\tfrac{1}{2}\theta = \sqrt{1-\sin^2\theta} = \frac{2\sqrt{Rr}}{R+r},$$

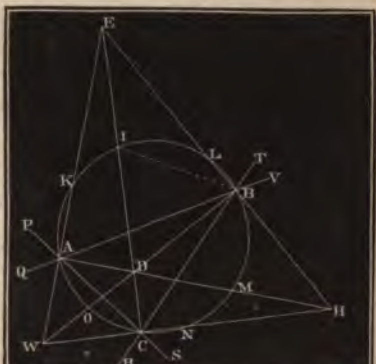
$$\therefore \sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = \frac{4(R-r)\sqrt{Rr}}{(R+r)^2}.$$

This is substantially the solution given by all the competitors.

*Prize Solution of Problem V.*

"Four circles may be described, each of which shall touch the three sides of a triangle, or those sides produced. If six straight lines be drawn, joining the centres of these circles two and two, prove that the middle points of these six lines are in the circumference of the circle circumscribing the given triangle."

Let $ABC$ be the given triangle. Bisect its angles by the lines $AH$, $BW$, $CE$. These lines contain the centres of the tangent circles, since each line bisects the angle formed by two tangents. $D$ is the centre of the inscribed circle, and the centres of the three other circles, $H$, $W$, and $E$ are determined by drawing $HW$, $WE$, and $EH$ perpendicular respectively to $CE$, $AH$, and $BW$; for by this construction $HW$, $WE$, and $EH$ are made to bisect, respectively, the angles $BCS$, $CAQ$, and $ABT$ formed by tangents to the circles, and therefore contain the centres of the circles. Circumscribe about the triangle the circle $AIBNO$, and draw $IB$.



The angle $IB\,O$ is measured by $\frac{1}{2}(IA + A\,O)$; also the angle $IDB$ is measured by $\frac{1}{2}(IB + O\,C)$. But $IB = IA$ and $OC = AO$.

$\therefore$ the angle $IB\,O = IDB$ and their complements $EBI = IEB$.

$\therefore$ the triangles $BDI$ and $EIB$ are isosceles, and $EI = IB = ID$.

Similarly it may be proved that $MH = MD$ and $W\,O = O\,D$. Again, from the secants $A\,E$ and $E\,C$, $\frac{E\,I}{EK} = \frac{E\,A}{E\,C}$, and from the similar triangles $E\,A\,D$ and $E\,C\,W$, $\frac{E\,A}{E\,D} = \frac{E\,C}{EW}$. Combining these two proportions, $\frac{EI}{ED} = \frac{EK}{EW}$ or $\frac{EI}{ED - EI} = \frac{EK}{EW - EK} \therefore \frac{EI}{ID} = \frac{EK}{KW}$; but $EI = ID \therefore EK = KW$.

Similarly it may be proved that $WN = NH$ and $HL = LE$. Hence the middle points of the lines connecting the centres of the four tangent circles are in the circumference of the circumscribing circle.

This solution is by Mr. George A. Osborne, Jr. Several other solutions of this interesting problem are also of decided excellence; and it is only for want of room in the Monthly that we do not recommend them for publication. The analytical solutions of Messrs. George B. Hicks and George W. Jones, although long, are of a high order of merit.

No complete sets of solutions of the Prize Problems in the second number of the Monthly have been received; and none of the competitors are entitled to a prize.

Joseph Winlock,
Chauncey Wright,
Truman Henry Safford.

## NOTE ON THE PROPOSITION OF PYTHAGORAS.

By Rev. A. D. Wheeler, Brunswick, Maine.

The truth of this proposition may be shown mechanically, by means of very simple apparatus. Two methods are here presented.

1. Cut from wood or pasteboard four equal right-angled triangles; and three squares, corresponding to the three sides of one of those triangles. They may be disposed as in the figures below, and these figures are manifestly equal. Hence the larger square must be equal to the sum of the other two.

$$a^2 + b^2 + 2ab = c^2 + 2ab,$$
$$\therefore a^2 + b^2 = c^2.$$

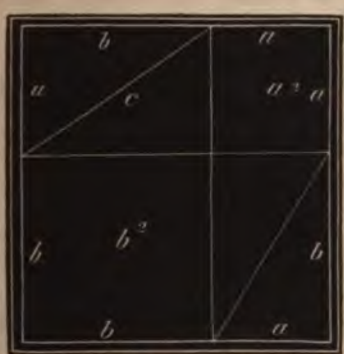



Fig. 1.


Fig 2.

2. From any suitable material, cut out two squares, joined at one of their sides, as in Fig. 3. From these squares cut off the triangles $ABC$, $BDE$, as in Fig. 4, making $AB = b$; whence there will remain $BD = a$. We have then the squares upon the base and perpendicular. Let the triangle $ABC$ turn on a hinge at $C$, until it comes into the position $CHF$, and the triangle $BDE$ turn on a hinge at $E$ until it comes into the position $EGF$. We

have then the square on the hypothenuse, which must of course be equal to the sum of the other two squares, since it is constructed out of them.

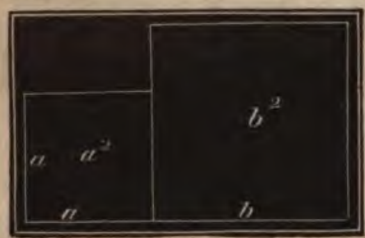



Fig. 3.

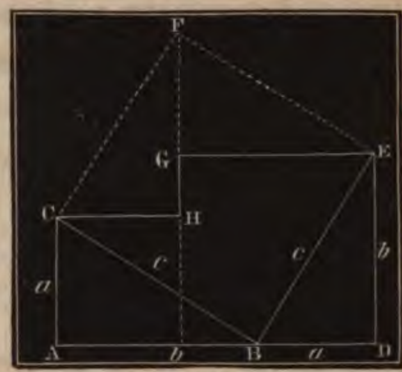



Fig. 4.

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## NOTE ON THE INTERPRETATION OF ALGEBRAIC RESULTS.

BY W. H. PARKER,
Professor of Mathematics in Middlebury College, Vermont.

In the discussion of algebraic problems, we sometimes find the result *zero,* arising from particular suppositions made upon the quantities that enter into the value of $x$.

What is the proper interpretation of this result? At the risk of seeming presumptuous in discussing a question which already has the seal of authority upon it, I will attempt to answer it in part, by considering two cases which arise in discussing the following familiar problem.

Problem. Upon a line on which two lights are placed, whose intensities at the distance 1, and whose distance apart, are given; it is required to find the point which is equally illuminated by them, assuming that the intensity of the same light, at different distances, varies inversely as the squares of the distances.


Let $A\,B$, the distance between the lights, be represented by $c$;

and the distance of the required point from $A$ considered as the origin of distances, by $x$. Let $a =$ the intensity of the light $A$ at the distance 1; and $b =$ the intensity of $B$ at the same distance. Then $\frac{a}{x^2} =$ the intensity of $A$ at the required point; and $\frac{b}{(c-x)^2} =$ the intensity of $B$ at the same point. Since the intensities are equal at that point, we have the equation $\frac{a}{x^2} = \frac{b}{(c-x)^2}$; whence $x = \frac{c\sqrt{a}}{\sqrt{a} \pm \sqrt{b}}$.

If we suppose $c = 0$, and $a > b$, or $a < b$, both values of $x$ reduce to 0. How shall we interpret this? We are told, it shows that the points of equal illumination coincide with the one where the lights are placed. If $a > 2b$, obviously the interpretation would not be different.

This conclusion appears to me unsound. Let us examine it. If the lights are placed at $A$, and $AS$ be a unit of distance,

A $\frac{1}{4}$ $\frac{1}{2}$ S

then at $\frac{1}{2}$ $AS$ the intensities are $\left(\frac{a}{(\frac{1}{2})^2}\ \frac{b}{(\frac{1}{2})^2} =\right)$ $4a$ and $4b$; at $\frac{1}{4}$ the distance $AS$ the intensities are $16a$, $16b$; at $\frac{1}{10}$ the distance $100a$, $100b$, and so on till we reach the limit which we are approaching, that is, the point where the lights are placed. By inspecting the expressions for the intensities, as we approach the point $A$, $4a$, $4b$; $16a$, $16b$; $100a$, $100b$, . . . . . we see that their relative intensity is the same throughout. If $a = 100b$ at the distance $AS$, it $= 100b$ at the distance 0; which shows that when the lights are unequal, and occupy the same point on the line, that point cannot be as much illuminated by one light as by the other.

The unsoundness of the conclusion may be exhibited in another way. Let it be required in the problem to find the point which is *twice* as much illuminated by one light as by the other. The values of $x$ now become $\frac{c\sqrt{a}}{\sqrt{a} \pm \sqrt{2b}}$. If we suppose $c = 0$, and $a > 2b$, or

$a < 2b$, both values of $x = 0$; which shows (following the interpretation adopted in the previous case) that the points, which are twice as much illuminated by one light as by the other, coincide with the point where the lights are placed.

In the former problem, when we were seeking the point of *equal* illumination, we found, even when $a > 2b$, that the point where the lights are placed is equally illuminated by them; now we find, when $a > 2b$, that the point where the lights are placed is twice as much illuminated by one as by the other. That is, the point is at the same time both equally illuminated by the two, and twice as much illuminated by one as by the other. A process which leads to these contradictory results must of course be at fault.

Shall we conclude, therefore, that algebraic reasoning is not always to be relied upon? By no means. Since all the transformations in algebraic equations are made by the use of axioms, no logical process is more simple or more reliable. If our primitive equation be true, all the resulting equations are necessarily true.

*The error to which I have invited attention is one of *interpretation.* We call to our aid the algebraic process, to determine the distance of the required point from the fixed point $A$. Algebra did not assume to prove that there is such a point (that was taken for granted in the very enunciation of the problem); but, if there be such a point, to determine its distance from the origin $A$. Now the result 0 shows that there is no such point on either side of the lights, and *shows nothing more.* Its language is "no distance." It affirms nothing. It merely denies; and denies only in respect to every other point on the line. To the question, is that point equally illuminated? it gives no answer. As in the discussion of a general problem, each new supposition converts it into a new and particular problem; it may happen that some of these will contain impossible conditions. The one we have been considering is of

* See p. 178.

this character. This we have already proved by the application of our assumed physical law, which shows, that, when the lights are together, if their intensities are unequal at any point on the line, they are unequal at every point.

The other case proposed is when $c = 0$, and $a = b$. Here the first value of $x = 0$. This is said to show that the point occupied by the lights is equally illuminated by them. But it seems to me simply to deny that the point of equal illumination between the lights is anywhere else; without affirming the existence of such a point. "No distance" is its language here, as in the former case. When by an inspection of the problem, or by a formal application of our physical law, we find the new problem possible, then the algebraic result shows the position of the point.

If, then, *zero* affirms nothing in regard to the possibility or impossibility of the conditions of the question, and (like infinity) is sometimes the answer to an impossible problem; when we come to interpret such a result, we are not to proceed upon the assumption that the thing required in the problem is possible, as we do when the result is a real and finite quantity; but are first to determine, by considering the nature of the question, whether the conditions are possible or not, and interpret the algebraic result accordingly.

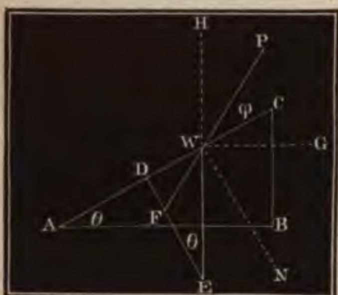
## THEORY OF THE INCLINED PLANE, FOR ELEMENTARY INSTRUCTION.

BY THOMAS SHERWIN,
Principal of the English High School, Boston, Mass.

Let a heavy body, which we call $W$, of inappreciably small magnitude, be placed upon the inclined plane $A\,C$ at the point $W$, and suppose that it is kept in equilibrium by a power $P$ acting in the direction $WP$, $AB$ being the horizontal base, and $B\,C$ the

height of the plane. We are to investigate the relations of the power, weight, and pressure upon the plane.

Let $\theta$ be the angle of elevation of the plane, and $\varphi$ the angle which the direction of the power makes with the plane. The weight acting vertically, through $W$ draw an indefinite vertical line, and take on it $WE$ equal to as many linear units as the weight contains units of weight. Through $E$ draw $ED$ perpendicular to $AC$, and produce $PW$ until it meets $ED$ in $F$. The weight, represented in quantity and direction by $WE$, may, by the principles of the resolution of force, be resolved into two other forces, represented in quantity and direction by $WF$ and $FE$, the two other sides of the triangle $WFE$. The force $WF$, acting directly opposite to the power, must, in case of equilibrium, be equal to the power, and the force $FE$, acting perpendicularly to the plane, produces pressure which is resisted by the plane. Call this pressure $p$. The power, weight, and pressure are then represented respectively by $WF$, $WE$, and $FE$.




Hence, $P:W = WF:WE = \sin FEW:\sin WFE$.

But the triangles $ABC$ and $DWE$ are similar, since each has a right angle, and since $WE$ is parallel to $BC$. Therefore the angle $FEW = \theta$; and $\sin WFE = \sin WFD = \cos FWD = \cos \varphi$. Therefore,

(1) $$P:W = \sin\theta : \cos\varphi.$$

Again, $P:p = WF:FE = \sin\theta : \sin FWE$. But $FWE = DWE - FWD = 90° - \theta - \varphi = 90° - (\theta + \varphi)$; $\therefore \sin FWE = \sin[90° - (\theta + \varphi)] = \cos(\theta + \varphi)$. Hence,

(2) $$P:p = \sin\theta : \cos(\theta + \varphi).$$

Since the antecedents are alike in (1) and (2), the cosequents are proportional; $\therefore$

(3) $$W : p = \cos\varphi : \cos(\theta + \varphi).$$

The three proportions given above are general; that is, they are applicable, whatever angle the direction of the power may make with the plane.

Suppose, now, that the power acts parallel to the plane. In this case $\varphi$ is zero, and (1), (2), and (3) become

(4) $P : W = \sin\theta : \cos 0 = \sin\theta : R = BC : AC$;

(5) $P : p = \sin\theta : \cos\theta = BC : AB$;

(6) $W : p = \cos 0 : \cos\theta = R : \cos\theta = AC : AB.$

Hence, when the power acts parallel to the plane, *the power is to the weight as the sign of elevation is to radius, or as the height of the plane is to the length; the power is to the pressure as the sine of elevation is to its cosine, or as the height of the plane is to its base; the weight is to the pressure as radius is to the cosine of elevation, or as the length of the plane is to its base.*

If the power acts parallel to the base, $\varphi$ becomes equal in value to $\theta$; but it is negative, since it is reckoned below the line $WC$. Hence, $\varphi = -\theta$, and (1), (2), and (3) become

(7) $P : W = \sin\theta : \cos(-\theta) = \sin\theta : \cos\theta = BC : AB$;

(8) $P : p = \sin\theta : R = BC : AC$;

(9) $W : p = \cos\theta : R = AB : AC.$

Hence, when the power acts parallel to the base, *the power is to the weight as the sine of the elevation is to its cosine, or as the height of the plane is to the base; the power is to the pressure as the sign of elevation is to radius, or as the height of the plane is to the length; the weight is to the pressure as the cosine of the elevation is to radius, or as the base of the plane is to the length.*

Observe that, whether the power acts parallel to the plane or parallel to the base, the power corresponds to the height, and the weight corresponds to that part of the plane parallel to which the power acts.

Suppose $\varphi$ equal to the complement of $\theta$; then (1) becomes $P : W = \sin\theta : \sin\theta$; and the power and weight are equal, as they evidently should be, since the power then acts vertically upwards. Likewise (2) becomes

$$P : p = \sin\theta : 0 \therefore p = \frac{P \times 0}{\sin\theta} = 0.$$

If the power acts downward and perpendicularly to $A\,C$, $\varphi = -\,90°$, and (1) becomes

$$P : W = \sin\theta : 0 \therefore P = \frac{W \times \sin\theta}{0} = \text{infinity}; \text{ and (3) gives}$$

$$W : p = 0 : \sin\theta \therefore p = \frac{W \times \sin\theta}{0} = \text{infinity}.$$

Thus it appears that the power, when it acts perpendicularly to the plane, is infinite, and that, if such a power were applied, the pressure would also be infinite. But the expression for the power is, in this case, to be regarded rather as a symbol of impossibility, for no two forces acting at right angles to each other can be in equilibrium.

The limits of possibility, consistent with equilibrium, for the direction of the power, are vertically upwards, and at right angles to the plane downwards. The direction may reach the former limit, and approach indefinitely near to the latter. The angle which these limiting directions make with each other is evidently the supplement of the elevation. Thus, $H\,W\,N = 180° - \theta$.

By examining proportions (1), (4), and (7), we see, since no cosine except that of zero can be so great as radius, that the ratio of the power to the weight is least when the power acts parallel to the plane. The power, therefore, acts most advantageously in this direction. This truth is manifest independently of analysis, since the power acts directly opposite to that part of the weight which tends to move the body down the plane.

If in (4) we call $\theta$ zero, we have

$$P : W = 0 : R \therefore P = \frac{W \times 0}{R} = 0,$$

and the same supposition in (6) gives

$$W:p = R:R;$$

showing that, in this case, the power is zero, and that the pressure is equal to the weight, as they manifestly should be, since the plane is horizontal.

If in the same proportions we suppose $\theta = 90°$, we have

$$P:W = R:R, \text{ and } W:p = R:0 \ \therefore p = \frac{W \times 0}{R} = 0.$$

Hence, in this case, the power and weight are equal, and the pressure is zero.

It follows also from (4) and (6), that, since the sine increases and the cosine decreases with the increase of the angle up to 90°, the ratio of the power to the weight is less, and the ratio of the pressure to the weight greater, the less the elevation of the plane. For the sake of simplicity we have supposed the weight to be of infinitesimal magnitude. But all that has been demonstrated is applicable to a body of any magnitude, provided the weight acts at the point $W$, and the power acts in a direction which would pass through the centre of gravity of the body.

## NOTE ON TWO NEW SYMBOLS.

BY BENJAMIN PEIRCE,
Professor of Mathematics in Harvard College, Cambridge, Mass.

The symbols which are now used to denote the Neperian base and the ratio of the circumference of a circle to its diameter are, for many reasons, inconvenient; and the close relation between these two quantities ought to be indicated in their notation. I would propose the following characters, which I have used with success in my lectures: —

 to denote ratio of circumference to diameter,

 to denote Neperian base.

It will be seen that the former symbol is a modification of the letter *c* (*circumference*), and the latter of *b* (*base*).

The connection of these quantities is shown by the equation,

$$\text{Ⴖ}^{\text{Ⴖ}} = (-1)^{-\sqrt{-1}}.$$

## THE NOTATION OF ANGLES.

By James Mills Peirce, Cambridge, Mass.

Professor Peirce, in his work on Analytic Mechanics, has introduced a method of denoting angles by writing the letters which represent the sides, one above the other. Thus, the angle between the axes of $x$ and $y$ in a rectilinear coördinate-system is denoted by the symbol $\frac{y}{x}$, which may be read $x$-$y$.

This method seems to answer, in the fullest possible manner, the purposes of a notation. It is at once simple and expressive. Each symbol, being determined by a principle and not chosen arbitrarily, carries its meaning on its face; and it is of course desirable that this should be true, as far as possible, of all notation, so that, in using general formulæ, we may not be under the necessity of looking up the significations of the symbols which they involve.

This system of notation may be somewhat further developed, and it will then be found to have other advantages besides those which have been pointed out.

1. The system may be made universal in its application by using *Greek letters* to denote the *directions* of lines, without reference to their length. Thus, if $\varrho$ denotes the axis in a system of polar coördinates, the polar angle will be $\frac{r}{\rho}$.

2. This notation affords a distinction between the two opposite circular directions which may be supposed to belong to the same angle. If a line be supposed to revolve about the point $O$, in the

accompanying figure, from the position $\alpha$ to the position $\beta$, its amount of rotation will be measured by the angle $\frac{\beta}{\alpha}$; but if it revolve from $\beta$ to $\alpha$, its rotation will be measured by $\frac{\alpha}{\beta}$. Since these rotations are equal in amount, but opposite in direction,


$$\tfrac{\beta}{\alpha}=-\tfrac{\alpha}{\beta};$$

that is, inverse angular symbols are negatives of each other.

3. By means of this notation, angles may be added by a mere inspection of the forms of their symbols. Thus we may write

$$\tfrac{\varepsilon}{\delta}+\tfrac{\gamma}{\varepsilon}+\tfrac{\alpha}{\gamma}+\tfrac{\beta}{\alpha}=\tfrac{\beta}{\delta};$$

for this is only saying that if a line rotate from $\delta$ to $\varepsilon$, then from $\varepsilon$ to $\gamma$, thence to $\alpha$, and thence to $\beta$, the resultant rotation is measured by the angle $\frac{\beta}{\delta}$. Again, by § 2,

$$\tfrac{\gamma}{\varepsilon}-\tfrac{\gamma}{\alpha}-\tfrac{\delta}{\varepsilon}+\tfrac{\beta}{\alpha}=\tfrac{\varepsilon}{\delta}+\tfrac{\gamma}{\varepsilon}+\tfrac{\alpha}{\gamma}+\tfrac{\beta}{\alpha}=\tfrac{\beta}{\delta}.$$

Hence, if a polynomial which consists of angular symbols can be so arranged, that, when all its terms are made positive, the upper letter of the first term is the lower letter of the second, the upper of the second the lower of the third, &c., the polynomial is equivalent to the angle made by the upper line of the last term with the lower line of the first. The same principle may be used to *decompose* angles.

This proposition is identical with HAMILTON'S Theorem of Versions. (Quaternions, Arts. 49, 65, &c.)

4. It is no objection to the rule of § 3, that it leaves a doubt as to whether the angles are to be measured in the most natural manner; that is, so as to be less than 180°. This ambiguity does not arise from the notation, but is inherent in the very notion of an angle, which may always have any one of an infinite series of values, differing by 360°. When, however, an angle is

treated through its trigonometric functions, this ambiguity may be disregarded.

5. Some of the above remarks may be illustrated by a solution of the problem, *To transform from one system of rectangular coördinates in a plane to another.*

Let the old system be that of $x, y$; let the new system be that of $x_1, y_1$; and let the coördinates of the new origin, referred to the old system, be $x^\circ$ and $y^\circ$.

Then, by § 3, $\genfrac{}{}{0pt}{}{y_1}{x} = \genfrac{}{}{0pt}{}{x}{x} + \genfrac{}{}{0pt}{}{y_1}{x_1} = \frac{1}{2}\pi$[*] $+ \genfrac{}{}{0pt}{}{x_1}{x}$.

The projections of $x_1$ and $y_1$ on the axis of $x$ are respectively

$$x_1 \cos \genfrac{}{}{0pt}{}{x_1}{x}, \quad y_1 \cos\left(\tfrac{1}{2}\pi + \genfrac{}{}{0pt}{}{x_1}{x}\right) = -y_1 \sin \genfrac{}{}{0pt}{}{x_1}{x};$$

and their projections on the axis of $y$ are respectively

$$x_1 \sin \genfrac{}{}{0pt}{}{x_1}{x}, \quad y_1 \sin\left(\tfrac{1}{2}\pi + \genfrac{}{}{0pt}{}{x_1}{x}\right) = y_1 \cos \genfrac{}{}{0pt}{}{x_1}{x}.$$

Hence the projections of the broken line formed by $x_1$ and $y_1$ on the axes of $x$ and $y$ are respectively

$$x - x^\circ = x_1 \cos \genfrac{}{}{0pt}{}{x_1}{x} - y_1 \sin \genfrac{}{}{0pt}{}{x_1}{x},$$
$$y - y^\circ = x_1 \sin \genfrac{}{}{0pt}{}{x_1}{x} + y_1 \cos \genfrac{}{}{0pt}{}{x_1}{x};$$

and we have

$$x = x^\circ + x_1 \cos \genfrac{}{}{0pt}{}{x_1}{x} - y_1 \sin \genfrac{}{}{0pt}{}{x_1}{x},$$
$$y = y^\circ + x_1 \sin \genfrac{}{}{0pt}{}{x_1}{x} + y_1 \cos \genfrac{}{}{0pt}{}{x_1}{x}.$$

6. The proposed symbol gives no more difficulty in printing than an ordinary fraction. For instances in which it occurs elevated or depressed out of the line, see PEIRCE's Analytic Mechanics, pp. $52_{13,\,31}$, $53_{2,\,5,\,8}$, $101_{3}$, &c., &c.

This notation is recommended to the attention of mathematicians.

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[*] See Prof. PEIRCE's Note on page 167. Of the desirableness of especial symbols to denote the quantities named, there can be but little doubt; and those suggested possess one essential requisite of a good notation, facility of use, as Prof. PEIRCE's experience in the lecture room proves. Besides, as obvious modifications of $c$ and $b$, they can be easily distinguished and remembered. We hope to see them exclusively adopted in the Monthly. For the advantages of this "Notation of Angles," see the valuable work on Analytic Geometry by the Author of this paper. — ED.

## MATHEMATICAL PRINCIPLES OF DIALING.

BY GEORGE EASTWOOD,
Assistant in the Office of the American Ephemeris and Nautical Almanac, Cambridge, Mass.

MY object in preparing these papers on Dialing is simply to bring into the small space of a few pages what the student might otherwise be obliged to seek through many volumes. This being my sole aim, and claiming nothing for these papers on the score of originality, I have not thought it necessary to give credit even in those cases where there is no doubt about the authorship.

### I. HORIZONTAL DIALS.

(1) It is not proposed in this place to enter into a history of Sun-dials. The invention of more exact and more accurate methods of measuring time, for all practical and scientific purposes, has, in a great measure, superseded their use, and deprived them of much of that interest and importance which were once ascribed to them. But, although their utility has been superseded by clocks and watches, the mathematical principles upon which they were and may be constructed remain unimpaired, and are eminently calculated to amuse and instruct the aspiring student.

(2) The following definitions ought not to be lost sight of:—

A *horizontal dial* is one that is traced on a horizontal plane.

A *vertical dial* is one that is constructed on a vertical plane. It may be *east*, *west*, *north*, or *south*, according to the cardinal point which it may face.

*Vertical declining dials* do not face any one cardinal point.

*Oblique dials* are those constructed on planes which make oblique angles with the horizon. They have the name of *reclining dials* when they lean backwards from the observer, and *proclining* when they project forward.

An *equinoctial dial* is that whose plane is perpendicular to the earth's axis, or parallel to the equator.

The *declination* of a plane is an arc of the horizon comprised between the plane and the plane of the prime vertical.

The *azimuth* of a plane is the arc of the horizon comprised between the plane and the plane of the meridian, and is the complement of the declination.

The *meridian* of a plane is *that* meridian plane which is perpendicular to the plane of the dial. This plane differs from the meridian of the place, the latter being always perpendicular to the horizon.

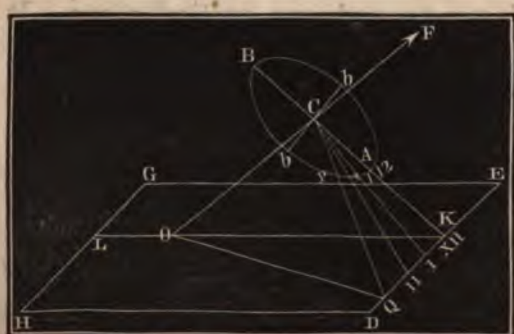
The *substyle* of a dial is the common section of its plane and the plane of its meridian, or it is the projection of the *style* of the dial upon its plane. In horizontal and in vertical south and north dials, the substyle coincides with the twelve o'clock hour line, but not in declining dials.

The *difference of longitude* of a dial plane is the angle which the plane of its meridian makes with the meridian of the place.

The *latitude* of a dial plane is the angle which the axis makes with the plane; this is the latitude of the place where the dial would be a horizontal one.

The *style* or *axis* of a dial must always point to the pole of the heavens.

(3) The equinoctial dial being the simplest of all dials to construct, and the horizontal dial ranking next to it, we will begin our investigations with the latter.




In the annexed diagram, then, let $G\,E\,D\,H$ be a horizontal plane, on which a dial is to be traced; $L\,O\,K$ a meridian line, $O\,C\,F$ a straight line or rod in the plane of the meridian, pointing to the pole, and making with $O\,K$ the angle $F\,O\,K$, equal to the latitude of the dial. Suppose $B\,A\,P$

to be an equinoctial dial, $O\,C\,F$ its axis, and $C$ its centre. Produce $C A$, the meridian line on this dial, to meet $L K$ in $K$. For obvious reasons, the plane of the shadow will turn uniformly about the axis $O\,C\,F$, meeting the equinoctial plane in some line $C P Q$, and the horizontal plane in an analogous line $O\,Q$. Let $C_1$, $C_2$, &c., be the hour lines after noon on the equinoctial dial, and $O\,I$, $O\,I\,I$, &c., the corresponding lines on the horizontal dial; the former will make, with the meridian line $C\,A\,K$, angles proportional to the time from noon, and will be known when the hour is given, 15° being counted to the hour. Suppose now the plane of the equinoctial dial to be extended till it meets the horizontal plane in the line $Q\,K$; this line of intersection of the two planes is obviously perpendicular to the meridian lines $C\,K$, $O\,K$. The problem to be resolved is, therefore, to find the hour angle $K\,O\,Q$ at the centre of the horizontal dial, corresponding to any given angle $K\,C\,Q$ at the centre of the equinoctial dial, which measures the time from noon.

Let $h =$ hour angle $K\,C\,Q$ on the equinoctial dial,

$h' =$ hour angle $K\,O\,Q$ on the horizontal dial;

then the right triangles $C\,K\,Q$, $O\,K\,Q$ give

$$K\,Q = C\,K \tan h,$$
$$K\,Q = O\,K \tan h'.$$

Put angle $K\,O\,C = \beta =$ latitude of the place for which the dial is to be constructed; then, $C$ being a right angle, we have

$$C\,K = O\,K \sin\beta,$$
$$\therefore \tan h' = \tan h \sin\beta,$$

is the general equation of the hour angles on a horizontal dial, for any latitude.

(4) If the dial, instead of being horizontal, be required to be a vertical north or south dial, a very slight consideration will convince the young student, that, if the vertical south dial were carried to a

place whose latitude is the complement of the given latitude, it would be a horizontal one for that place. The equation of the hour angles on a vertical south or north dial, for latitude $\beta$, is, therefore,

$$\tan h' = \tan h \cos \beta .$$

(5) Vertical east and west dials are described on vertical planes which coincide with the meridian plane. The style or axis of a vertical east or west dial is parallel to its plane at any given height above it. As the plane of its shadow turns uniformly about its axis, it will cut off from the line, which is perpendicular to the six o'clock hour line, distances which will be the tangents of the angles generated by the shadow; that is, the tangents of the hour angles from six. If therefore $d =$ height of the style, $h =$ the hour angle from six, and $h' =$ distance cut off by the shadow, then

$$h' = d \tan h .$$

(6) And if $h$ be taken for the hour angle from the twelve o'clock hour line, then

$$h' = d \tan h$$

will answer for a polar dial.

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## NOTE FROM G. W. HILL, ESQ., TO THE EDITOR.

In Mr. Watson's article in the January Number of the Monthly, on the curve of a drawbridge, I would like to notice that the investigation could be much shortened. For, drawing vertical lines from the roller $E$ and centre of gravity of the platform, along which lines the weights $W_1$ and $W$ tend to move, and applying the principle of virtual velocities we have

$$W_1 \delta \left(a - r \cos \varphi\right) - W \delta \left(\frac{l \cos \theta}{2}\right) = 0 ;$$

or

$$W_1 \delta \left(r \cos \varphi\right) + W \delta \left(\frac{l\left((c - r)^2 - 2a^2\right)}{4a^2}\right) = 0 .$$

By integrating

$$W_1 r \cos \varphi + W \frac{l((c-r)^2 - 2a^2)}{4a^2} = \text{constant};$$

or, since $$\frac{2 W_1 a^2}{Wl} = B,$$

$$2 B r \cos \varphi - 2 c r + r^2 = \text{constant},$$

which is the equation to the curve.

*Rutger's College*, Jan. 22, 1859.

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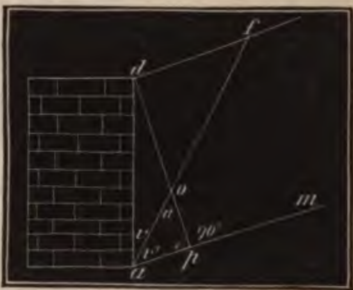
## ON THE HORIZONTAL THRUST OF EMBANKMENTS.

By Capt. D. P. Woodbury, U. S. Corps of Engineers.

What is the horizontal thrust against a vertical wall of an embankment of homogeneous cohesive earth rising to an indefinite plane parallel to the natural slope?




Let $F$ = that thrust.

Let $df$, parallel to the natural slope $a\,m$, be the indefinite surface of the embankment.

Let $F'$ = the variable horizontal force which, acting at right angles to $a\,d$, shall be just sufficient to prevent the descent of any prism $a\,d\,f$ along its base $f\,a$. $Q$ = the surface $a\,d\,f$.

Let angle $d\,a\,f = v$; angle $d\,a\,m = a = 90° -$ the angle of friction.

From $d$, the top of the wall and foot of the slope, let $dp$ be drawn perpendicular to the natural slope, $a\,m$. We shall find

$$F = \tfrac{1}{2}\,(dp)^2 = \tfrac{1}{2}\,h^2 \sin^2 a; \qquad (h = a\,d)$$

and the prism of greatest thrust will be the trapezoid resting on $a\,d$, and bounded by the indefinite lines $a\,m$ and $df$. This anomalous result may be thus proved.

The variable forces $F'$ and $Q$ may be resolved into the components $F' \cos v$, $Q \sin v$, perpendicular to $af$, and $F' \sin v$, $Q \cos v$, parallel to $af$. The force $F' \sin v$, acting along $af$ upwards, must, assisted by the friction due to normal pressure on $af$, just equal the parallel and opposite force $Q \cos v$; that is,

(1) $$F' \times \sin v + (F' \cos v + Q \sin v) f = Q \cos v.$$

For $f$ substitute $\cot a = \frac{\cos a}{\sin a}$, and reduce. There results

(2) $$F' = Q \times \operatorname{tang}(a - v).$$

(3) But $$Q = \tfrac{1}{2}\, ad \times df \times \sin a = \tfrac{1}{2}\, h^2 \times \frac{\sin a \sin v}{\sin(a - v)},$$

which gives $$F' = \tfrac{1}{2}\, h^2 \frac{\sin a \sin v}{\cos(a - v)}.$$

Let $v' = a - v$; then $a - v' = v$,

$$F' = \tfrac{1}{2}\, h^2 \sin a\, (\sin a - \cos a \operatorname{tang} v')$$

(4) $$= \tfrac{1}{2}\, h \sin a\, (h \sin a - h \cos a \operatorname{tang} v')$$

$$= \tfrac{1}{2}\, dp\, (dp - op) = \tfrac{1}{2}\, dp \times do.$$

The maximum value of $F' = F$ evidently corresponds to that direction of $af$ in which $op$ is zero, or $do = dp$, hence

(5) $$F = \tfrac{1}{2}\, h^2 \sin^2 a = \tfrac{1}{2}\, (dp)^2.$$

The thickness of a rectangular wall able to withstand this thrust, the curve of pressure crossing the base at one third its length from the exterior edge, is given by the equation ($e$ = thickness of wall),

(6) $F \times \frac{1}{3} h = eh \times \frac{1}{6} e$; giving $e^2 = 2F$, $e = h \sin a = dp$.

We have thus far supposed the density of the earth and of the wall to be the same, that is, unity. Let $\omega$ = the weight of a cubic unit of the earth; $\omega'$ = the weight of a cubic unit of the wall. We have

(7) $$F = \tfrac{1}{2}\, \omega \times (dp)^2; \text{ and } e = dp \sqrt{\frac{\omega}{\omega'}}.$$

The rule of the French engineers, which consists in doubling the horizontal thrust and determining the thickness of pile on the con-

dition that the resultant of the thrust thus increased, and the weight of the wall, shall pass through the exterior edge of the base, gives

$$2\,F \times \tfrac{1}{3}\,h = \tfrac{1}{2}\,h\,e^2;\ \text{or}\ e = dp\,\sqrt{\tfrac{2}{3}};$$

or, introducing $\omega$ and $\omega'$, as above,

$$e = dp\,\sqrt{\frac{2\,\omega}{3\,\omega'}}. \tag{8}$$

The ratio of $e$ in (8) and (7) is $\sqrt{\tfrac{2}{3}} = .8164$. That is, the "rule" does not give a sufficient thickness. Formula (4) includes the case of a perfect fluid; in which $a$ being 90°, $\cos a$ is zero, and $\cos a \times \tan v' = o\,p = 0$, whatever be the angle $v'$ or $v$.

The pressure of a perfect fluid therefore upon a vertical surface, which, the density being unity, is known to be $\tfrac{1}{2}\,h^2$, is only a particular case of a general law.

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[From WALTON'S Collection of Mechanical Problems.]

## PROBLEM IN PROJECTILES.

By Prof. LONGFELLOW and WILLIAM WALTON.

SWIFT of foot was Hiawatha;
He could shoot an arrow from him,
And run forward with such fleetness,
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward,
Shoot them with such strength and swiftness,
That the tenth had left the bow-string
Ere the first to earth had fallen.

Supposing Hiawatha to have been able to shoot an arrow every second, and, when not shooting vertically, to have aimed so that the flight of the arrow might have the longest range, prove that it would have been safe to bet long odds on him if entered for the Derby.

## EDITORIAL REMARKS ON PROF. PARKER'S "NOTE ON THE INTERPRETATION OF ALGEBRAIC RESULTS."

Although we cannot entirely agree with Prof. Parker in his conclusions, still the question admits of discussion, as his interesting article clearly shows. Nor is Prof. Parker the only one who has ever doubted the correctness of the usual interpretation of the case in question, given by Bourdon and the many others who have used the "Problem of the Lights," or the entirely similar problem of "Points of equal Attraction," in the discussion of equations of the second degree. We know that many teachers, as well as students, have questioned it; and, what is more, those authors who have omitted the case altogether in their discussion of the problem, pretty evidently did not feel entirely satisfied with it. We are therefore much obliged to Prof. Parker for his clear and pointed statement of the issue, and beg to append the following remarks.

We remark, first, that if the origin be taken at some point $O$ instead of $A$, and we put $OA = \alpha$,

$O \qquad\qquad A \qquad\qquad B$

$OB = \beta$, and let $x$ denote the distance from $O$ to the points of equal illumination; then if $a$ and $b$ denote the respective intensities of the lights $A$ and $B$, we have

(1) $$\frac{a}{(x - \alpha)^2} = \frac{b}{(\beta - x)^2}; \text{ or}$$

(2) $$(\beta - x)^2\, a = (x - \alpha)^2\, b; \text{ or}$$

(3) $$x = \frac{\beta\sqrt{a} \pm \alpha\sqrt{b}}{\sqrt{a} \pm \sqrt{b}}.$$

If now we suppose that $\beta = \alpha$, that is, that the lights are together at $A$, then

$$x = \frac{\alpha\,(\sqrt{a} \pm \sqrt{b})}{\sqrt{a} \pm \sqrt{b}} = \alpha;$$

and both roots $= \alpha$ instead of *zero*, to which they would be equal

if the origin were taken at the point occupied by the lights. If, therefore, the particular hypothesis introduces impossible conditions into the problem, the fact, if indicated at all by any peculiarity of the roots, must be indicated by the presence of *equal* roots, and not because they both happen to be zero for a special origin. But the equal roots do not of themselves necessarily indicate any impossibility, and it is only by observing that the equation itself becomes impossible that we detect the fact that the hypothesis has introduced impossible conditions. When $\beta = \alpha = x$, we obtain

$$0^2 \times a = 0^2 \times b,$$

and the roots $\alpha$ satisfy form (2), whatever be the intensities; but the members of the equation in form (1) become unequal infinities for unequal intensities, and this is where the symbol of impossibility appears.

The fallacy consists in multiplying both sides of equation (1) by factors which become zero for the particular hypothesis; and failing to attend to this consideration led BOURDON into the oversight.

Again, if we put $y = \frac{a}{(x - \alpha)^2}$ and $y' = \frac{b}{(\beta - x)^2}$, and construct the curves corresponding to these equations, it is plain that their intersections will correspond to all possible solutions of (1). The intensity curve of the light $A$ consists of two branches, to both of which the ordinate through the light is an asymptote; the axis of $x$ is also an asymptote to both branches. The same remark applies to the intensity curve of the light $B$. The right hand branch of $A$'s curve will cut both branches of $B$'s, and these two are the only points of intersection and the only solutions.

When $\beta = \alpha$, the ratio of the ordinates or intensities is

$$\frac{y}{y'} = \frac{a\,(\alpha - x)^2}{b\,(x - \alpha)^2} = \frac{a}{b} = \text{constant}$$

for all values of $x$, including $x = \alpha$; and therefore, when both

lights occupy the same point, and $B$'s intensity at a unit's distance is less than $A$'s at the same distance, $B$'s intensity curve will fall entirely within $A$'s, and there is no intersection and no solution. When, therefore, two lights of unequal intensities occupy the same place, there is no point in space which they equally illuminate; not even the one in which they are both situated.

When the intensities are equal, that is, when $a = b$, as well as $\beta = \alpha$, the curves coincide throughout their whole extent, and this indeterminateness indicates that all points on the line are equally illuminated by the lights. But so long as there is any distance at all, however small, between the lights, the curves will cut in two points, and the problem will have two possible solutions.

Finally, we remark that the error in question does not seem to us to be one of interpretation, as Prof. PARKER supposes. It consists not in an impossible form of the roots, but in an *implicit fallacy*, to which algebraic transformations are often liable.

We must conclude, that when $x = 0$ is the true root of an equation, it does not, like infinity, indicate impossible conditions in the problem, but must be interpreted in the usual manner.

## THEOREM ON RECTANGULAR COÖRDINATES.

BY W. P. G. BARTLETT,
Nautical Almanac Office, Cambridge, Mass.

THE well-known equation,

$$\cos {}^{x'}_{x} = \cos {}^{y'}_{y} \cos {}^{z'}_{z} - \cos {}^{y'}_{z} \cos {}^{z'}_{y},$$

may be simply deduced as follows. Let the systems of axes intersect a sphere at $x\,y\,z$ and $x'\,y'\,z'$, the origin of each system being placed at the centre of the sphere. Let the quadrantal spherical

triangle $x\,y\,z$ be moved on the surface of the sphere into the position $x\,y_0\,z_0$, then in the same way into the position $x_0\,y_0\,z'$, and finally into the position $x'\,y'\,z'$. Let $y'\,p$ be a great circle arc perpendicular to the arc $z\,y$. Then from the various right angled triangles thus formed we get, by considering the arcs, the following relations between the angles which they represent.

$$
\begin{aligned}
\cos\tfrac{x'}{x} &= \cos\tfrac{x}{x_0}\cos\tfrac{x'}{x_0} = \cos\tfrac{z'}{z_0}\cos\tfrac{y'}{y_0} = \cos\tfrac{z'}{z_0}\cos\tfrac{p}{y_0}\cos\tfrac{y'}{p} \\
&= \cos\tfrac{z'}{z_0}\cos\tfrac{y'}{p}\cos\left(\tfrac{y}{p}\backsim\tfrac{y}{y_0}\right) = \cos\tfrac{z'}{z_0}\cos\tfrac{y'}{p}\left(\cos\tfrac{y}{p}\cos\tfrac{y}{y_0} + \sin\tfrac{y}{p}\sin\tfrac{y}{y_0}\right) \\
&= \cos\tfrac{z'}{z_0}\cos\tfrac{y'}{p}\left(\cos\tfrac{y}{p}\cos\tfrac{z}{z_0} - \cos\tfrac{z}{p}\cos\tfrac{y}{z_0}\right) \\
&= \cos\tfrac{y'}{p}\cos\tfrac{y}{p}\cos\tfrac{z'}{z_0}\cos\tfrac{z}{z_0} - \cos\tfrac{y'}{p}\cos\tfrac{z}{p}\cos\tfrac{z'}{z_0}\cos\tfrac{y}{z_0} \\
&= \cos\tfrac{y'}{y}\cos\tfrac{z'}{z} - \cos\tfrac{y'}{z}\cos\tfrac{z'}{y}.
\end{aligned}
$$

I have used a German notation, $\backsim$, to denote the difference between $\tfrac{y}{p}$ and $\tfrac{y}{y_0}$, because $p$ may fall on the other side of $y_0$. Although I have not been able to find this demonstration, I should be surprised if it had not been given before. Can any one furnish information on the point?

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# Mathematical Monthly Notices.

*Asteroids for the Year* 1859. *A Supplement to the American Ephemeris for* 1861.

Besides the opposition Ephemerides which have been published from time to time, by the Superintendent of the American Ephemeris and Nautical Almanac for comparison with observations, this is the first regular issue of the Ephemerides of any considerable number of the Asteriod group as part of the American Ephemeris. The large number of the Asteroids already known, with their probable increase, will make the preparation of this department no insignificant part of the annual labor; and we propose therefore briefly to indicate the plan which has been adopted by Prof. Winlock for carrying on this part of the work. 1st. The same epoch and intervals of time are adopted in all the computations of Special Perturbations. 2d. The coördinates of the disturbing planets, and all that part of the labor independent of the particular Asteroid, are once for all carefully computed and checked for the adopted epoch and

intervals; thus saving the labor which must otherwise be performed for each separate Asteroid. 3d. Instead of the usual frequent correction of the elements, they will remain unchanged until the perturbations have accumulated to such a degree that it will be a saving of labor to incorporate them into a new set of corrected elements; and in the mean time any small corrections of the elements which may seem desirable will be combined with the perturbations. In this way a tolerably good set of elements will not probably need correction and change of epoch short of five thousand days. 4th. The computations of the Special Perturbations of all the Asteroids to which special methods are applied will be carried on simultaneously for the same dates, a labor-saving arrangement which no one but an experienced practical astronomer can fully appreciate. It gives us great pleasure to record this evidence of Prof. WINLOCK's able and judicious superintendence of the American Ephemeris.

This Supplement contains the Ephemerides of thirty-three of the fifty-six Asteroids, with Tables of elements and authorities. Almost the entire history of the Asteroid group, except a column of relative brightness to be added hereafter, is contained in the following pages, which we have thought it desirable to extract to save the readers of the Monthly the trouble of other reference; and for the same reason we add the following definitions of symbols. The Ecliptic is the plane of reference, which the plane of the orbit cuts in a straight line, called "The line of the Nodes."

The longitude of the Ascending node, or point through which the asteroid passes from south to north of the ecliptic, denoted by ☊, is the angular distance of this point from the vernal equinox, or first point of Aries.

The inclination of the plane of the orbit to that of the ecliptic is denoted by $i$, and the two elements ☊ and $i$ fix the position of the plane of the orbit

The longitude of the perihelion, denoted by $\pi$, fixes the position of the orbit in its own plane, and is counted on the Ecliptic from the first point of Aries to the ascending node, then on the plane of the orbit in the direction of the Asteroid's motion until we reach its perihelion, or point of least distance from the sun.

The mean distance from the sun is denoted by $a$, the eccentricity by $e$, and these two elements determine the size and shape of the orbit. The mean orbit longitude of the Asteroid for the epoch, denoted by L, is counted on the Ecliptic from the first point of Aries to the Ascending node, then on the plane of the orbit in the direction of the Asteroid's motion until we arrive at its place. The mean daily motion is denoted by $\mu$, which enables us to find the mean orbit longitude for any date before or after the epoch. If one imagines himself standing at the Sun on the north side of the Ecliptic, the angles are counted from right to left, that is, towards the east, and the motions of the Asteroids are in the same direction.

The Asteroids numbered (41) and (41)* have this interesting history: In 1856, May 23, Dr. Goldschmidt discovered Daphne. It was at this time near its conjunction, and was therefore lost in the sun's rays before any thing more than the first rough approximation to its orbit could be obtained. In 1857, September 9, he again observed what he supposed to be the lost Daphne; but Mr. Schubert has shown that the two sets of observations do not correspond to the same orbit, and therefore that the Daphne of 1856 is still a "missing star."

The following pages, on which are found the elements with authorities, are printed on duplicates of the electrotype plates prepared for this Supplement, for which we are indebted to the courtesy of Prof. WINLOCK.

① *Ceres.* — *Astronomical Journal*, Vol. III. p. 165, by Mr. ERNEST SCHUBERT, from a thorough discussion of observations from 1832 to 1853, taking account of perturbations by Jupiter only. They have been reduced by him from 1854, January 0, to 1859, September 7, by applying the perturbations depending on Jupiter and Saturn. Comparison with observations at opposition in 1858 gave $\Delta\,\alpha \cos\delta = -5''.2$, $\Delta\,\delta = +6''.2$.

② *Pallas.* — *English Nautical Almanac* for 1860, p. 572, by Mr. FARLEY, from eight oppositions, 1845 to 1853, inclusive, reduced, by addition of perturbations, depending on Venus, the Earth, Mars, Jupiter, and Saturn, to 1858, May 29, Greenwich. They nearly satisfy all the observations made at Greenwich near the times of oppositions as far as 1855 inclusive.

③ *Juno.* — *English Nautical Almanac* for 1859, p. 564, from twelve oppositions, 1841 to 1855 inclusive, reduced by addition of perturbations depending on Venus, the Earth, Mars, Jupiter, and Saturn. Comparison with Greenwich observations at opposition in 1856 gave $\Delta\,\alpha \cos\delta = -10''.7$, $\Delta\,\delta = +0''.7$, and at Königsberg in 1858,

$$\Delta\,\alpha \cos\delta = -21''.0, \quad \Delta\,\delta = +3''.0.$$

④ *Vesta.* — *English Nautical Almanac* for the year 1860, p. 575, by Mr. FARLEY, from twelve oppositions, 1840 to 1855 inclusive, reduced by addition of perturbations depending on Venus, the Earth, Mars, Jupiter, and Saturn. They very nearly satisfy all the observations made at Greenwich near the times of oppositions as far as 1855 inclusive, and observations at Königsberg in 1858, within about 5″.

⑤ *Astræa.* — *Berliner Astron. Jahrbuch* for the year 1858, by Professor ZECH. They have satisfied observations at seven oppositions, from 1845 to 1853 inclusive, and at the opposition in 1856 gave, about, $\Delta\,\alpha \cos\delta = +13''$, $\Delta\,\delta = +4''$.

⑥ *Hebe.* — *Astronomische Nachrichten*, Vol. XXXI. p. 13, by R. LUTHER, from four oppositions, 1847 – 1850; in 1857 the errors at opposition were $\Delta\,\alpha \cos\delta = +21''$, $\Delta\,\delta = -7''$.

⑦ *Iris.* — *Astronomische Nachrichten*, Vol. XXVIII. p. 277, by Mr. ERNEST SCHUBERT, from two oppositions, 1847 – 1848, reduced by addition of perturbations. They have agreed with observations since, until 1858, when the errors were

$$\Delta\,\alpha \cos\delta = 46'', \quad \Delta\,\delta = 15''.$$

⑧ *Flora.* — *Tables of Flora*, by Professor F. BRÜNNOW, Berlin, 1855. They were computed from four oppositions, 1848 – 1852.

⑨ *Metis.* — *Astronomische Nachrichten*, Vol. XXXVI. p. 71, by J. PH. WOLFERS, from six oppositions, 1848 – 1852. Have agreed with observations since; at opposition in 1857 the errors were $\Delta\,\alpha \cos\delta = -11''$, $\Delta\,\delta = -1''$.

⑩ *Hygea.* — *Astronomische Nachrichten*, Vol. XXXIX. p. 347, by Professor J. ZECH, from five oppositions, 1849 – 1854, reduced by addition of perturbations. At opposition in 1856 the errors were $\Delta\,\alpha \cos\delta = -8''$, $\Delta\,\delta = +1''$.

⑪ *Parthenope.* — *Astronomische Nachrichten*, Vol. XLI. p. 283, from four oppositions, 1850 – 1854. Errors in 1857, $\Delta\,\alpha \cos\delta = -3''$, $\Delta\,\delta = -6''$.

⑫ *Clio.* — *Astronomische Nachrichten*, Vol. XLV. p. 321, by Professor F. BRÜNNOW, from six oppositions, 1850 – 1856. Tables have been constructed by him.

⑬ *Egeria.* — *Astronomical Journal*, Vol. II. p. 282, by Professor J. S. HUBBARD, 1850 – 1851. Tables have been constructed by Professor PEIRCE.

(14) *Irene.* — *Astronomische Nachrichten*, Vol. XLII. p. 141, from four oppositions, 1851–1855, by C. BRUHNS. At opposition in November, 1857, the errors were

$$\Delta\alpha \cos\delta = -4'', \quad \Delta\delta = -1''.$$

(15) *Eunomia.* — *Astronomical Journal*, Vol. IV. p. 170, by Mr. ERNEST SCHUBERT, from four oppositions, 1851–1854. Have agreed well with observations since. At opposition in 1858 the errors were $\Delta\alpha \cos\delta = +3''$, $\Delta\delta = -3''$.

(16) *Psyche.* — Provisional elements selected, and reduced by Mr. SCHUBERT by addition of perturbations preparatory to a new determination of the orbit.

(17) *Thetis.* — *Berliner Astron. Jahrbuch*, 1859, p. 419, by E. SCHÖNFELD, from four oppositions, 1852–1856. The errors at opposition in 1857 were $\Delta\alpha \cos\delta = -38''$, $\Delta\delta = -13''$.

(18) *Melpomene.* — *Astronomical Journal*, Vol. V. p. 41, from four oppositions, 1852–1856. At opposition in 1858, $\Delta\alpha \cos\delta = +6''$, $\Delta\delta = -3''$.

(19) *Fortuna.* — *Astronomische Nachrichten*, Vol. XLVI. p. 247, by C. POWALKY, from four oppositions, 1852–1856. Errors at opposition in 1858, $\Delta\alpha \cos\delta = -10''$, $\Delta\delta = +5''$.

(20) *Massilia.* — *Astronomische Nachrichten*, Vol. XLV. p. 287, by W. GÜNTHER, from four oppositions, 1852–1856, perturbations by Jupiter alone being applied. In 1858,

$$\Delta\alpha \cos\delta = -11'', \quad \Delta\delta = +1''.$$

(21) *Lutetia.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 17, from four oppositions, perturbations by Jupiter alone being taken account of. Errors at opposition in 1858,

$$\Delta\alpha \cos\delta = +7'', \quad \Delta\delta = +1''.$$

(22) *Calliope.* — Vienna *Sitzungsberichte*, 1855, by Dr. C. HORNSTEIN, corrected by T. H. SAFFORD, Jr., so as to satisfy four oppositions, 1852–1856.

(23) *Thalia.* — *Astronomical Journal*, Vol. V. p. 107, by ERNEST SCHUBERT, from four oppositions, 1853–1856. Errors at opposition in 1858, $\Delta\alpha \cos\delta = +4''$, $\Delta\delta = +1''$.

(24) *Themis.* — *Astronomische Nachrichten*, Vol. XLVII. p. 161, by Dr. A. KRÜGER, from four oppositions, 1853–1856. At opposition in 1858, $\Delta\alpha \cos\delta = +2''$, $\Delta\delta = -1''$.

(25) *Phocæa.* — *Astronomische Nachrichten*, Vol. XLVI. p. 129, by W. GÜNTHER, from three oppositions, 1853–1856. Errors in 1857, $\Delta\alpha \cos\delta = +19''$, $\Delta\delta = -7''$.

(26) *Proserpina.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 171, by J. A. C. OUDEMANNS, corrected by M. HOEK to satisfy four oppositions, 1853–1857. Errors at the opposition in 1858, $\Delta\alpha \cos\delta = +14''$, $\Delta\delta = -2''$.

(27) *Euterpe* — *Astronomische Nachrichten*, Vol. XLVIII. p. 229, by W. GÜNTHER, from four oppositions, 1853–1858.

(28) *Bellona.* — *Berliner Astron. Jahrbuch*, 1859, from two oppositions, 1854–1855. They have not been compared with observations since.

(29) *Amphitrite.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 363, by W. GÜNTHER, from four oppositions, 1854–1858.

*Urania.* — *Astronomische Nachrichten*, Vol. XLVII. p. 21, by W. GÜNTHER, from ...itions, 1854–1857.

...*me.* — *Astronomische Nachrichten*, Vol. XLI. p. 289, by A. WINNECKE, from ...'854–1855. Errors at opposition in 1857 were $\Delta\alpha \cos\delta = +1''$, $\Delta\delta = +10''$.

(32) *Pomona.* — Elements selected and reduced by Mr. Ernest Schubert, preparatory to a new determination of the orbit.

(33) *Polyhymnia.* — Selected for corrrection by Mr. Schubert.

(34) *Circe.* — *Berliner Astron. Jahrbuch*, 1859, p. 420, from two oppositions, 1855 - 1856, by Dr. W. Klinkerfues. At opposition in 1857, the errors were,

$$\Delta\, a \cos \delta = -14'\, 3'', \quad \Delta\, \delta = -3'\, 45''.$$

(35) *Leucothea.* — Selected for correction by Mr. Ernest Schubert.

(36) *Atalanta.* — *Berliner Astron. Jahrbuch*, 1860, p. 404, from two oppositions, by Dr. W. Förster, 1855 - 1857; agreed well with observation in 1858.

(37) *Fides.* — *Astronomische Nachrichten*, Vol. XLV. p. 17, from one opposition, by G. Rümker, 1855 - 1856; in 1857 they were in error about 20″ in R. A. and 14″ in Dec.

(38) *Leda.* — *Berliner Astron. Jahrbuch*, 1860, from one opposition, 1856, by M. Löwy; agreed with observation in 1858 within about 2′ in R. A. and 1′ in Dec.

(39) *Lætitia.* — *Astronomische Nachrichten*, Vol. XLV. p. 379, from one opposition, 1856, by M. Allé.

(40) *Harmonia.* — *Astronomische Nachrichten*, Vol. XLIV. p. 281, from one opposition, 1856, by C. Powalky. Did not agree well with observation in 1857.

(41) *Daphne.* — *Astronomische Nachrichten*, Vol. XLVII. p. 26, from five days' observations by C. F. Pape, very uncertain.

(41)* *Astronomical Journal*, Vol. V. p. 174, by Mr. Ernest Schubert, from observations in 1857.

(42) *Isis.* — *Astronomische Nachrichten*, Vol. XLVI. p. 91, from observations in 1856. In December, 1857, the errors were $\Delta\, a = -1'.7$, $\Delta\, \delta = -0'.6$.

(43) *Ariadne.* — *Astronomische Nachrichten*, Vol. XLIX. p. 39, by E. Weiss, from observations in 1857.

(44) *Nysa.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 233, by M. Gussew, from observations in 1857.

(45) . . . . *Astronomische Nachrichten*, Vol. XLVIII. p. 359, by M. Löwy, from observations in 1857.

(46) *Hestia.* — *Astronomical Journal*, Vol. V. p. 153, by J. C. Watson.

(47) *Aglaia.* — From observations in 1857, by T. H. Safford, Jr. In February, 1858, the errors were $\Delta\, a = +50''$, $\Delta\, \delta = +20''$.

(48) *Doris.* — *Astronomische Nachrichten*, Vol. XLVII. p. 319, by C. Powalky.

(49) *Pales.* — *Astronomische Nachrichten*, Vol. XLVII. p. 315, by C. Powalky.

(50) *Verginia.* — *Astronomical Journal*, Vol. V. p. 118, by Mr. James Ferguson. They will probably give the place of the planet within 5′.

(51) *Nemausa.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 124, from a few observations, by Dr. W. Förster.

(52) *Europa.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 221, by Dr. H. S. Schultz. Approximate.

(53) *Calypso.* — *Astronomische Nachrichten*, Vol. XLVIII. p. 335, by W. Oeltzen.

(54) . . . . *Astronomische Nachrichten*, Vol. XLIX. p. 185, by Schjellerup.

(55) . . . . *Astronomical Journal*, Vol. V. p. 162, by T. H. Safford, Jr. and S. Newcomb.

| Symbol. | Name. | $\pi$. | $\Omega$. | $\phi$. | $i$. | $\mu$. | $L$. |
|---|---|---|---|---|---|---|---|
| | | ° ′ ″ | ° ′ ″ | ° ′ ″ | ° ′ ″ | ′ ″ | ° ′ ″ |
| 1 | Ceres. | 149 26 13.1 | 80 49 54.7 | 4 36 12.1 | 10 36 27.8 | 12 51.3333 | 346 48 15.4 |
| 2 | Pallas. | 122 7 38.4 | 172 38 32.7 | 13 50 57.1 | 34 42 29.8 | 12 49.4780 | 224 28 25.5 |
| 3 | Juno. | 54 0 55.8 | 170 58 22.0 | 14 50 35.7 | 13 3 9.8 | 13 33.8848 | 104 2 31.1 |
| 4 | Vesta. | 250 35 29.4 | 103 21 10.3 | 5 10 31.2 | 7 8 9.1 | 16 17.8432 | 218 26 1.1 |
| 5 | Astræa. | 134 35 35.7 | 141 24 48.5 | 10 57 8.3 | 5 19 35.2 | 14 17.9486 | 80 56 2.7 |
| 6 | Hebe. | 15 2 23.4 | 138 35 19.5 | 11 38 1.9 | 14 46 35.4 | 15 39.3481 | 124 54 18.6 |
| 7 | Iris. | 41 29 15.3 | 259 46 16.1 | 13 20 45.9 | 5 28 1.4 | 16 2.6335 | 322 34 38.8 |
| 8 | Flora. | 32 54 28.3 | 110 17 48.6 | 9 0 56.3 | 5 53 8.0 | 18 6.3310 | 68 48 32.0 |
| 9 | Metis. | 71 3 55.6 | 68 31 31.6 | 7 5 1.6 | 5 36 0.6 | 16 2.8856 | 128 8 12.7 |
| 10 | Hygea. | 227 47 58.8 | 287 38 34.2 | 5 46 16.6 | 3 47 9.3 | 10 34.8491 | 354 47 47.6 |
| 11 | Parthenope. | 316 10 7.1 | 125 3 41.1 | 5 40 30.3 | 4 36 57.9 | 15 23.7824 | 283 56 41.9 |
| 12 | Clio. | 301 39 24.7 | 235 34 41.7 | 12 38 44.1 | 8 23 19.4 | 16 35.8341 | 7 42 5.0 |
| 13 | Egeria. | 119 12 59.0 | 43 17 55.7 | 4 52 7.4 | 16 33 6.7 | 14 18.3861 | 138 44 42.6 |
| 14 | Irene. | 179 28 21.9 | 86 40 4.5 | 9 30 38.1 | 9 7 7.4 | 14 11.5608 | 67 12 20.6 |
| 15 | Eunomia. | 27 31 8.1 | 293 56 15.8 | 10 47 54.8 | 11 43 39.0 | 13 45.2229 | 238 54 5.1 |
| 16 | Psyche. | 13 16 14.8 | 150 35 34.0 | 7 42 49.7 | 3 4 6.5 | 11 50.0987 | 50 51 42.0 |
| 17 | Thetis. | 259 22 51.2 | 125 27 13.3 | 7 17 18.4 | 5 35 40.7 | 15 11.9760 | 210 1 24.3 |
| 18 | Melpomene. | 15 11 48.0 | 150 4 33.3 | 12 32 14.8 | 10 8 58.3 | 16 59.8395 | 304 33 25.3 |
| 19 | Fortuna. | 30 22 50.2 | 211 29 28.7 | 9 5 10.8 | 1 32 28.8 | 15 30.1578 | 148 28 55.8 |
| 20 | Massilia. | 98 28 37.6 | 206 41 27.6 | 8 15 42.3 | 0 41 7.3 | 15 48.7396 | 195 16 53.9 |
| 21 | Lutetia. | 327 2 45.2 | 80 27 23.3 | 9 19 32.1 | 3 5 11.1 | 15 33.5610 | 41 24 9.0 |
| 22 | Calliope. | 58 16 41.1 | 66 36 54.7 | 5 56 53.6 | 13 44 51.9 | 11 54.9070 | 76 59 2.0 |
| 23 | Thalia. | 123 58 40.6 | 67 38 34.4 | 13 23 56.7 | 10 13 13.6 | 13 52.4617 | 280 7 33.7 |
| 24 | Themis. | 137 54 9.7 | 36 10 30.3 | 6 44 53.0 | 0 49 1.8 | 10 34.6753 | 30 2 41.5 |
| 25 | Phocæa. | 302 46 9.0 | 214 4 54.6 | 14 37 38.8 | 21 35 53.6 | 15 53.6780 | 294 46 13.5 |
| 26 | Proserpina. | 235 17 26.8 | 45 53 14.6 | 5 1 15.7 | 3 35 40.3 | 13 39.6815 | 181 21 20.9 |
| 27 | Euterpe. | 87 39 0.0 | 93 44 45.0 | 9 57 22.5 | 1 35 31.1 | 16 26.6260 | 260 43 32.7 |
| 28 | Bellona. | 122 22 48.3 | 144 43 5.4 | 8 53 17.5 | 9 22 30.8 | 12 47.4862 | 159 3 36.8 |
| 29 | Amphitrite. | 56 39 6.6 | 356 26 51.8 | 4 9 3.1 | 6 7 49.6 | 14 28.8694 | 293 11 23.8 |
| 30 | Urania. | 31 23 24.7 | 308 13 46.3 | 7 18 22.7 | 2 5 56.9 | 16 16.0689 | 19 30 24.4 |
| 31 | Euphrosyne. | 93 51 6.6 | 31 25 23.0 | 12 28 29.8 | 26 25 12.4 | 10 32.8031 | 53 49 50.3 |
| 32 | Pomona. | 193 33 42.5 | 220 48 1.4 | 4 37 26.6 | 5 28 49.1 | 14 11.7238 | 134 30 20.0 |
| 33 | Polyhymnia. | 340 51 46.1 | 9 16 9.2 | 19 41 36.4 | 1 56 41.5 | 12 10.8833 | 266 47 55.8 |
| 34 | Circe. | 149 58 35.1 | 184 47 10.8 | 6 12 52.4 | 5 26 33.2 | 13 24.9883 | 193 36 37.2 |
| 35 | Leucothea. | 198 51 53.9 | 355 57 26.3 | 12 46 9.3 | 8 12 10.7 | 11 29.3084 | 173 36 11.3 |
| 36 | Atalanta. | 42 22 25.0 | 359 8 48.4 | 17 19 53.4 | 18 42 9.5 | 12 58.6000 | 36 19 53.2 |
| 37 | Fides. | 66 5 35.8 | 8 10 23.4 | 10 4 0.8 | 3 7 19.3 | 13 46.2860 | 42 34 30.3 |
| 38 | Leda. | 100 40 28.4 | 296 27 47.3 | 8 57 0.8 | 6 58 31.9 | 13 2.4484 | 112 55 7.2 |
| 39 | Lætitia. | 1 58 57.7 | 157 19 31.0 | 6 22 38.2 | 10 20 50.7 | 12 49.8940 | 146 44 19.7 |
| 40 | Harmonia. | 2 1 50.9 | 93 32 2.9 | 2 38 29.0 | 4 15 48.4 | 17 19.4100 | 222 12 9.1 |

| Symbol. | Period. | $a$. | $e$. | Epoch. | Date of Discovery. | By whom Discovered. |
|---|---|---|---|---|---|---|
| | d | | | | | |
| (1) | 1680.207 | 2.765938 | 0.080257 | 1859, Sept. 7.0000 | 1801, Jan. 1 | Piazzi, at Palermo. |
| (2) | 1684.258 | 2.770386 | 0.239367 | 1858, May 28.7860 | 1802, Mar. 28 | Harding, at Göttingen. |
| (3) | 1592.365 | 2.668678 | 0.256176 | 1858, Jan. 28.7860 | 1804, Sept. 1 | Olbers, at Bremen. |
| (4) | 1325.366 | 2.361339 | 0.090204 | 1858, April 22.7860 | 1807, Mar. 29 | Olbers, at Bremen. |
| (5) | 1510.580 | 2.576500 | 0.189992 | 1849, Dec. 30.7488 | 1845, Dec. 8 | Hencke, at Driessen. |
| (6) | 1379.680 | 2.425418 | 0.201657 | 1857, Feb. 12.7488 | 1847, July 1 | Hencke, at Driessen. |
| (7) | 1346.307 | 2.386147 | 0.230832 | 1858, July 18.7488 | 1847, Aug. 13 | Hind, at London. |
| (8) | 1193.007 | 2.201386 | 0.156704 | 1848, Jan. 0.7488 | 1847, Oct. 18 | Hind, at London. |
| (9) | 1345.954 | 2.385730 | 0.123321 | 1858, June 29.7488 | 1848, April 25 | Graham, at Markree. |
| (10) | 2041.430 | 3.149373 | 0.100557 | 1851, Sept. 16.7488 | 1849, April 12 | De Gasparis, at Naples. |
| (11) | 1402.928 | 2.452588 | 0.098887 | 1858, June 26.7488 | 1850, May 13 | De Gasparis, at Naples. |
| (12) | 1301.423 | 2.332811 | 0.218920 | 1850, Dec. 30.7488 | 1850, Sept. 13 | Hind, at London. |
| (13) | 1509.810 | 2.575625 | 0.084873 | 1851, Dec. 5.0000 | 1850, Nov. 2 | De Gasparis, at Naples. |
| (14) | 1521.912 | 2.589368 | 0.165230 | 1857, Nov. 19.7488 | 1851, May 20 | Hind, at London. |
| (15) | 1570.486 | 2.644180 | 0.187357 | 1859, May 11.0000 | 1851, July 29 | De Gasparis, at Naples. |
| (16) | 1825.098 | 2.922752 | 0.134225 | 1860, Nov. 20.0000 | 1852, Mar. 17 | De Gasparis, at Naples. |
| (17) | 1421.090 | 2.473710 | 0.126865 | 1856, April 3.7488 | 1852, April 17 | Luther, at Bilk. |
| (18) | 1270.788 | 2.296060 | 0.217078 | 1859, July 2.0000 | 1852, June 24 | Hind, at London. |
| (19) | 1393.312 | 2.441368 | 0.157922 | 1858, Mar. 2.7488 | 1852, Aug. 22 | Hind, at London. |
| (20) | 1366.023 | 2.409386 | 0.143696 | 1858, April 20.7488 | 1852, Sept. 19 | Chacornac, at Marseilles. |
| (21) | 1388.232 | 2.435431 | 0.162045 | 1853, Jan. 1.7488 | 1852, Nov. 15 | Goldschmidt, at Paris. |
| (22) | 1439.977 | 2.495579 | 0.103630 | 1852, Dec. 30.7488 | 1852, Nov. 16 | Hind, at London. |
| (23) | 1556.829 | 2.628824 | 0.231732 | 1859, July 10.0000 | 1852, Dec. 15 | Hind, at London. |
| (24) | 2041.989 | 3.149947 | 0.117504 | 1856, Sept. 24.7488 | 1853, April 5 | De Gasparis, at Naples. |
| (25) | 1358.949 | 2.401060 | 0.252533 | 1857, July 9.7488 | 1853, April 6 | Chacornac, at Marseilles. |
| (26) | 1581.102 | 2.656079 | 0.087521 | 1857, Mar. 19.7488 | 1853, May 5 | Luther, at Bilk. |
| (27) | 1313.568 | 2.347305 | 0.172896 | 1859, June 13.7488 | 1853, Nov. 8 | Hind, at London. |
| (28) | 1688.630 | 2.775177 | 0.154507 | 1854, Feb. 27.7488 | 1854, May 1 | Luther, at Bilk. |
| (29) | 1491.594 | 2.554866 | 0.072383 | 1859, July 8.7488 | 1854, Mar. 1 | Luther, at Bilk. |
| (30) | 1327.805 | 2.364199 | 0.127174 | 1858, Oct. 8.7488 | 1854, July 22 | Hind, at London. |
| (31) | 2048.030 | 3.156158 | 0.216013 | 1854, Dec. 30.7488 | 1854, Sept. 1 | Ferguson, at Washington |
| (32) | 1521.620 | 2.589039 | 0.080617 | 1860, Jan. 24.7488 | 1854, Oct. 26 | Goldschmidt, at Paris. |
| (33) | 1773.197 | 2.867075 | 0.336087 | 1858, April 13.7488 | 1854, Oct. 28 | Chacornac, at Paris. |
| (34) | 1609.961 | 2.688302 | 0.108253 | 1855, April 9.4488 | 1855, April 15 | Chacornac, at Paris. |
| (35) | 1880.145 | 2.981229 | 0.221025 | 1860, Feb. 14.0000 | 1855, April 19 | Luther, at Bilk. |
| (36) | 1664.526 | 2.748705 | 0.297900 | 1855, Dec. 30.7488 | 1855, Oct. 5 | Goldschmidt, at Paris. |
| (37) | 1563.465 | 2.641907 | 0.174798 | 1855, Dec. 30.7488 | 1855, Oct. 5 | Luther, at Bilk. |
| (38) | 1656.339 | 2.739685 | 0.155576 | 1855, Dec. 30.7488 | 1856, Jan. 12 | Chacornac, at Paris. |
| (39) | 1683.349 | 2.769387 | 0.111075 | 1855, Dec. 31.7488 | 1856, Feb. 8 | Chacornac, at Paris. |
| (40) | 1246.861 | 2.267148 | 0.046085 | 1856, June 30.7488 | 1856, Mar. 31 | Goldschmidt, at Paris. |

| Symbol. | Name. | π. | ☊. | φ. | i. | μ. | L. |
|---|---|---|---|---|---|---|---|
| | | ° ′ ″ | ° ′ ″ | ° ′ ″ | ° ′ ″ | ′ ″ | ° ′ ″ |
| (41) | Daphne. | 230 21 29.8 | 180 5 50.8 | 11 40 57.0 | 15 48 23.0 | 15 54.1100 | 202 28 48.5 |
| (41)* | . . . . . . | 303 17 28.1 | 195 29 38.4 | 11 42 3.8 | 7 38 19.1 | 14 40.0100 | 335 48 51.5 |
| (42) | Isis. | 317 57 48.4 | 84 27 49.7 | 12 52 50.1 | 8 34 39.6 | 15 34.4490 | 276 45 1.9 |
| (43) | Ariadne. | 277 14 9.5 | 264 29 27.4 | 9 38 46.6 | 3 27 47.6 | 18 4.5177 | 224 5 10.4 |
| (44) | Nysa. | 111 46 12.3 | 130 54 33.4 | 8 25 51.6 | 3 41 56.6 | 15 36.4700 | 232 55 23.7 |
| (45) | . . . . . . | 235 4 34.4 | 147 51 37.7 | 4 54 10.7 | 6 35 59.1 | 13 5.1037 | 215 29 8.3 |
| (46) | Hestia. | 355 4 36.8 | 181 26 43.6 | 9 45 28.8 | 2 17 48.3 | 14 36.5246 | 333 1 31.1 |
| (47) | Aglaia. | 314 16 26.4 | 4 29 19.6 | 7 21 42.5 | 5 0 24.7 | 12 5.8040 | 0 37 45.4 |
| (48) | Doris. | 77 11 47.7 | 185 13 39.9 | 4 25 19.8 | 6 29 44.0 | 10 47.9290 | 359 3 37.3 |
| (49) | Pales. | 32 49 23.3 | 290 27 1.0 | 13 44 54.4 | 3 8 25.0 | 10 54.4680 | 10 29 28.9 |
| (50) | Verginia. | 10 29 59.0 | 173 30 22.8 | 16 41 14.6 | 2 47 45.7 | 13 42.0410 | 12 5 7.9 |
| (51) | Nemausa. | 190 12 40.0 | 175 37 43.9 | 3 36 13.0 | 10 14 39.4 | 16 7.6380 | 172 47 1.8 |
| (52) | Europa. | 102 10 43.7 | 129 55 43.8 | 5 52 11.5 | 7 23 48.7 | 10 50.6371 | 147 35 49.8 |
| (53) | Calypso. | 94 38 52.3 | 143 30 27.8 | 10 23 3.6 | 5 3 38.8 | 14 0.0860 | 169 59 43.1 |
| (54) | . . . . . . | 306 19 28.9 | 313 22 43.9 | 10 50 23.7 | 11 31 21.0 | 13 9.0720 | 329 25 3.3 |
| (55) | . . . . . . | 21 47 23.8 | 10 51 28.2 | 7 41 19.4 | 7 36 47.4 | 12 46.0760 | 10 49 0.2 |

## PERIODIC COMETS.

| Name. | π. | ☊. | φ. | i. | μ. | L. |
|---|---|---|---|---|---|---|
| | ° ′ ″ | ° ′ ″ | ° ′ ″ | ° ′ ″ | ′ ″ | ° ′ ″ |
| Halley's. | 304 32 16.6 | 55 10 43.7 | 75 19 40.2 | 162 14 54.9 | 0 46.5067 | 304 32 16.6 |
| Encke's. | 157 57 30.0 | 334 28 34.0 | (57 57 30.3) | 13 4 15.0 | 17 54.0500 | 157 59 18.0 |
| Biela's I. | 108 58 52.7 | 245 54 5.2 | 49 7 23.6 | 12 33 49.6 | 8 55.2767 | 108 58 52.7 |
| Biela's II. | 109 5 56.0 | 245 50 9.9 | 49 2 34.5 | 12 33 27.8 | 8 58.7065 | 109 5 56.0 |
| Faye's. | 49 49 4.6 | 209 45 23.4 | 33 42 43.4 | 11 21 36.7 | 7 55.1849 | 49 49 4.6 |
| Brorsen's. | 115 43 44.4 | 101 46 41.7 | 53 21 5.6 | 29 48 59.2 | 10 37.9355 | 115 43 44.4 |
| Winnecke's. | 275 59 53.0 | 113 0 53.1 | 47 35 5.2 | 10 42 43.4 | 11 48.0070 | 275 59 33.3 |
| Tuttle's. | 115 51 35.0 | 269 3 13.0 | 55 10 31.4 | 54 24 10.5 | 4 18.9576 | 116 10 44.5 |

| Symbol. | Period. | *a.* | *e.* | Epoch. | Date of Discovery. | By whom Discovered. |
|---|---|---|---|---|---|---|
| | d | | | | | |
| (41) | 1358.334 | 2.400337 | 0.202488 | 1856, May 31.2488 | 1856, May 23 | Goldschmidt, at Paris. |
| (41)* | 1472.710 | 2.533257 | 0.202805 | 1857, Sept. 16.2844 | 1857, Sept. 9 | Goldschmidt, at Paris. |
| (42) | 1386.913 | 2.433889 | 0.222920 | 1856, June 30.7488 | 1857, May 23 | Pogson, at Oxford. |
| (43) | 1195.001 | 2.203838 | 0.167565 | 1857, April 16.7488 | 1857, April 15 | Pogson, at Oxford. |
| (44) | 1383.921 | 2.430386 | 0.146618 | 1857, July 9.7488 | 1857, May 27 | Goldschmidt, at Paris. |
| (45) | 1659.191 | 2.742828 | 0.085469 | 1856, Dec. 30.7488 | 1857, June 27 | Goldschmidt, at Paris. |
| (46) | 1478.567 | 2.539968 | 0.169487 | 1857, Sept. 19.5000 | 1857, Aug. 16 | Pogson, at Oxford. |
| (47) | 1785.606 | 2.880435 | 0.128134 | 1857, Nov. 16.0000 | 1857, Sept. 15 | Luther, at Bilk. |
| (48) | 2000.219 | 3.106845 | 0.077105 | 1857, Oct. 30.7488 | 1857, Sept. 19 | Goldschmidt, at Paris. |
| (49) | 1980.234 | 3.086115 | 0.237660 | 1857, Oct. 30.7488 | 1857, Sept. 19 | " " |
| (50) | 1576.563 | 2.650994 | 0.287150 | 1857, Oct. 5.0000 | 1857, Oct. 4 | Ferguson, at Washington. |
| (51) | 1339.344 | 2.377912 | 0.062853 | 1858, Mar. 2.3400 | 1858, Jan. 22 | Laurent, at Nismes. |
| (52) | 1991.893 | 3.098218 | 0.102270 | 1858, Mar. 3.2052 | 1858, Feb. 4 | Goldschmidt, at Paris. |
| (53) | 1542.700 | 2.612894 | 0.180250 | 1858, April 27.2635 | 1858, April 4 | Luther, at Bilk. |
| (54) | 1642.436 | 2.724332 | 0.188066 | 1858, Sept. 25.1342 | 1858, Sept. 11 | Goldschmidt, at Paris. |
| (55) | 1691.750 | 2.778581 | 0.133791 | 1858, Sept. 27.3496 | 1858, Sept. 11 | Searle, at Albany. |

## PERIODIC COMETS.

| Period. | *a.* | *e.* | Epoch. | Perihelion Passage. |
|---|---|---|---|---|
| d | | | | |
| 27866.953 | 17.988470 | 0.967391 | 1835, Nov. 15.6941 | 1912.1 |
| 1206.648 | 2.218135 | 0.847663 | 1858, Oct. 18.2488 | 1862.1 |
| 2421.174 | 3.528733 | 0.756119 | 1852, Sept. 22.7316 | 1859.4 |
| 2405.760 | 3.513750 | 0.755201 | 1852, Sept. 23.4975 | 1859.4 |
| 2727.360 | 3.820286 | 0.555020 | 1858, Sept. 12.3908 | 1866.2 |
| 2031.554 | 3.139206 | 0.802313 | 1857, Mar. 20.0128 | 1862.8 |
| 1830.490 | 2.928505 | 0.738276 | 1858, May 2.2488 | 1863.3 |
| 5004.680 | 5.726007 | 0.820904 | 1858, Feb. 27.7488 | 1871.9 |

# Editorial Items.

Asaph Hall, Esq., Assistant at Harvard College Observatory, says: "I have found the following erratum in 'Theoria Motus,' Art. 83, page 108 of Davis's translation. In the denominator of the right-hand member of the last equation in the Article, for cot $\frac{1}{2}(N''-N)$ read cot $\frac{1}{2}(N''-N')$. In an erratum previously sent by Mr. Hall, $\nu$ is misprinted for $\chi$. ..... George W. Jones, of the Senior Class of Yale College, solved all the prize problems in the October number of the Monthly, instead of George W. Fisher, whose name was printed by mistake...... It gives us pleasure to add the following names to our list of co-operators and contributors: — W. LeRoy Brown, Principal of Bloomfield Academy, Joy Depot, Albemarle Co., Va. Gerardus B. Docharty, LL.D., Professor of Mathematics in the New York Free Academy. J. H. Good, Professor of Mathematics in Heidelberg College, Tiffin, Seneca Co., Ohio. John F. Lanneau, of Furman University, Greenville, S. C...... In our last Number we gave the Prize Problem in the "Lady's and Gentleman's Diary" for 1859, hoping that some of our readers might feel sufficient interest in it to compete for the prize. Mr. Simon Newcomb, Assistant upon the American Ephemeris, and Mr. George B. Vose, Assistant in the U. S. Coast Survey, Washington, D. C., have sent us solutions, and we should like permission from these gentlemen to communicate their solutions to Prof. W. S. B. Woolhouse, the Editor of the Diary...... Note from Rev. T. W. Higginson to the Editor: —

"Worcester, Mass., January 14, 1859.

"Dear Sir: — The enclosed note found its way into print without my intending or expecting it; but I thought you might like to see it, as evidence of the value which we, in this direction, attach to your Magazine. I wish that the same thing could be done in other places; for I am satisfied that there are in many of our High Schools (certainly in this one), young mathematicians, both male and female, who can solve the easier portion of your prize problems.

"Truly yours,

"T. W. Higginson."

"Liberal offer to the High School. — We have been permitted to publish the following letter, addressed to Mr. Calkins, Principal of the Mathematical Department of the High School. It is gratifying to record such evidence of the interest our distinguished citizens are taking in this noble science: —

'Worcester, January 9, 1859.

'Dear Sir: — You have no doubt observed the prize problems in the Mathematical Monthly, and have formed an opinion as to whether your pupils could perform many of them. If you think it worth while, I would gladly promise a bound copy of Vol. I. of the Monthly to the member of your school sending the best solutions of the largest number of problems between this time and the close of the volume. This might be an additional incentive to some who distrusted their ability to gain Mr. Runkle's prizes.

'It would be an especial gratification to me if I could thus secure some female names among the monthly lists of those solving the problems.

'Cordially yours,

'T. W. Higginson.'"

This evidence of interest in the Monthly is deeply gratifying, and we are ready to coöperate in all possible ways to carry out the suggestion of Mr. HIGGINSON. If the teachers of Mathematics in our High Schools and Academies think our Prize Problems are too difficult, and will communicate those they think of the proper degree of difficulty, we will gladly insert five of them each month, entitled *Prize Problems for Students in Academies, High and Normal Schools.* The teachers and friends of each Institution can then offer such prizes, with such conditions, as they see fit. It would, however, be exceedingly desirable for each Prize Committee to report to us at the end of the year; as we should then be able to show the relative standard of such institutions in regard to mathematical instruction, as well as announce the name of the student who had solved the largest number of the prize problems.

BOOKS RECEIVED. — Cours de Mécanique Appliquée; par M. MAHISTRE. Paris: Mallet-Bachelier, 1858. — Lecons de Mécanique Élémentaire a l'usage Des Candidats à l'École Polytechnique et à l'École Normale supérieure; par M. OSSIAN BONNET. Première Partie. Paris: Mallet-Bachelier, 1858. — Encyclopédie Mathématique ou exposition complète de toutes des Mathématiques d'après les principes de la Philosophie des Mathématiques de Hoëné Wronski; par A. S. DE MONTFERRIER. Tomes 1, 2, 3. To be continued in monthly parts. — Physique a l'usage des Gens du Monde, 308 Magnifiques Vignettes. 12 mo.; par GANOT. Cours de Physique de l'École Polytechnique. 8vo. JARMIN. — Construction of Wrought-Iron Bridges; by LATHAM. — Manual of applied Mechanics; by RANKINE. — Quarterly Journal of Pure and Applied Mathematics for Nov. 1858. — No. 1, of Vol. III. Nouvelles Annales de Mathématiques, Decembre, 1858.

MATHEMATICAL PAPERS *published by the Smithsonian Institution, in the " Smithsonian Contributions to Knowledge."*

Vol. II. Article I. Researches relative to the Planet Neptune; by SEARS C. WALKER, Esq., pp. 60.

Appendix I. Ephemeris of the Planet Neptune for the date of the Lalande observations of May 8 and 10, 1795, and for the oppositions of 1846, 1847, 1848, and 1849, pp. 32. II. Ephemeris of the Planet Neptune for the years 1850 and 1851; by SEARS C. WALKER, Esq., pp. 20. III. Occultations visible in the United States during the year 1851; computed by JOHN DOWNES, Esq., pp. 26.

Vol. III. Article II. Researches on Electrical Rheometry; by A. SECCHI, pp. 60, and three plates.

Appendix I. Ephemeris of the Planet Neptune for the year 1852; by SEARS C. WALKER, Esq., pp. 10. II. Occultations visible in the United States, and other parts of the world during the year 1852; computed by JOHN DOWNES, Esq., pp. 36.

Vol. VIII. Article IV. The Tangencies of Circles and of Spheres; by BENJAMIN ALVORD, Major U. S. Army, pp. 16, and nine plates.

Vol. IX. Article II. On the Relative Intensity of the Heat and Light of the Sun upon different latitudes of the earth; by L. W. MEECH, pp. 58, and six plates.

Appendix. New Tables for determining the values of the Coefficients in the Perturbative Function of Planetary Motion, which depend upon the ratio of the mean distances; by JOHN D. RUNKLE, Assistant in the office of the American Ephemeris and Nautical Almanac, pp. 64. Asteroid Supplement to New Tables for determining the values of $b_s^{(i)}$ and its derivatives; by JOHN D. RUNKLE, pp. 72.

Besides the above, the Appendices to Vols. VIII. and IX. contain a list of the " Publications of Learned Societies and Periodicals in the Library of the Smithsonian Institution. Part I.

pp. 40; part II. pp. 38." These Appendices show the very great value of this department of the Library, and it is gratifying to observe that almost the whole of this invaluable collection has been donated to the Smithsonian Institution by the various Societies.

MATHEMATICAL PAPERS *published by the American Academy of Arts and Sciences, in the New Series, commenced in* 1833.

Vol. II. Article IV. The latitude of the Cambridge Observatory, in Massachusetts, determined from transits of stars over the prime vertical observed during the months of December, 1844, and January, 1845, by William C. Bond, A. A. S., Major James D. Graham, A. A. S., and George P. Bond; by BENJAMIN PEIRCE, A. A. S.

Vol. III. Article VI. Some Methods of computing the Ratio of the Distances of a Comet from the Earth; by GEORGE P. BOND, Assistant at the Cambridge Observatory.

Vol. IV. Article VII. On some Applications of the Method of Mechanical Quadratures; by GEORGE P. BOND.

Vol. V. Article VII. On the Rings of Saturn; by GEORGE P. BOND.

Vol. VI. Part I. On the Use of Equivalent Factors in the Method of Least Squares; by GEORGE P. BOND.

There are many other papers in these volumes of great value, but of a less decided mathematical character, which we do not include for want of space.

## Obituary.

*Decease of Professor* WILLIAM CRANCH BOND, *the distinguished Director of the Observatory of Harvard College, aged* 69 *years.*

THIS melancholy event occurred on the afternoon of Saturday, January 29, at the Observatory. An affection of the heart, with which he had been afflicted for several years, finally terminated in his sudden, but not unexpected, death. Thus has passed away one of the most eminent and successful cultivators of Astronomical Science. His various papers, published in the "Memoirs of the American Academy of Arts and Sciences," the various "Astronomical Journals," the "Annals of the Astronomical Observatory of Harvard College," his many brilliant discoveries in Astronomy, the invention of the "Spring Governor" for recording astronomical observations by means of electro-magnetism, are among the monuments of the ability, skill, and industry which have marked his honorable career. And his claims to distinction have been recognized by his contemporaries. In 1842 Harvard College gave him the honorary degree of Master of Arts; he was a Member of the American Academy of Arts and Sciences, Member of the American Philosophical Society of Philadelphia, and of the National Institute at Washington, Associate of the Royal Astronomical Society of London, Corresponding Member of the Institute of France, the Philomathic Society of Paris, the Accademia de' Nuovi Lincei at Rome, the Society of Natural Sciences at Cherbourg, and also of other learned societies.

# THE
# MATHEMATICAL MONTHLY.

Vol. I....MARCH, 1859....No. VI.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

Any side of a triangle is cut in the ratio of $m$ to $n$, and the line joining this point to the opposite vertex is cut in the ratio of $m+n$ to $l$; to find the coördinates of the point of section.

### II.

Can the equations $y^2+x=a$, $x^2+y=b$ be solved by quadratics?

### III.

Find the polar equation of the line passing through the points of which the polar coördinates are $r'$, $\varphi'$; $r''$, $\varphi''$.

### IV.

Find the condition that $Ax+By+C=0$ should be tangent to $(x-a)^2+(y-b)^2=r^2$.

### V.

Give the most concise method for finding the derivative of $a^x$.

The solutions of these problems must be received by the first of May, 1859.

## MATHEMATICAL HOLOCRYPTIC CYPHERS.

By PLINY EARLE CHASE, Philadelphia.

THE cyphers usually employed for secret correspondence, are formed by the mere substitution of symbols for the letters of the alphabet, one symbol only being appropriated to each letter. The comparative frequency with which the several letters recur in our language, is well known, therefore all cyphers of this kind are of little worth, as they may be easily translated by any expert into whose hands they may fall.

By the alternate use of two or three different symbols for each of the most common letters, the difficulty of translation is increased, and the value of the cypher proportionately enhanced. But even in this case the law is so simple, and the number of constants so great, that the key can generally be found after a few trials.

A holocryptic cypher should not only conceal the message that it is intended to convey, but it should also effectually hide the key to the message, which it can hardly do unless it destroys all symbolic uniformity, so that any given letter will be as likely to be represented by one symbol as by another. The various operations of Mathematics,—Addition, and the other simple rules of Arithmetic, Logarithms, Curvefunctions, &c., furnish an infinite variety of methods that will accomplish this end.

Let two correspondents, for instance, commence by arranging an alphabet in *rows* and *places*, in a manner similar to the following:—

| | | PLACES. | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| Rows. | 1 | X | U | A | C | O | N | Z | L | P | φ |
| | 2 | B | Y | F | M | & | E | G | J | Q | ω |
| | 3 | D | K | S | V | H | R | W | T | I | λ |

According to this alphabet the word "Philip" may be designated

by $\substack{133131\\959899}$, the upper line denoting the *row*, and the lower line the *place*, of the several letters.

If we multiply the lower line by 9, we obtain for our cypher $\substack{133131\\8639091} = Lnsi\varphi ix.$*

If the key that has been agreed upon, is the addition of 7854 (repeated as often as necessary), $959899 + 785478 = 1745377$. The left-hand figure of this sum may be rejected, as it will be readily known when the cypher is retranslated. We then obtain for our cypher $\substack{133131\\745377} = Zvhawz.$

If a logarithmic key has been adopted, we may obtain the following cypher: $\log 959899\dagger = 982229$; $\substack{133131\\982229} = Ptkukp.$

The following examples will illustrate the method of interpreting the three foregoing varieties of mathematical cypher.

| | | Key. |
|---|---|---|
| 1. | $E\omega ntxbxxdheokj = \substack{22131211332132\\60681111156528}$ | $\div$ 8 |
| 2. | $\Omega pcpwi\lambda maopiag = \substack{21113332111312\\09497904359937}$ | — 709 |
| 3. | $\Omega lgpntufkygl = \substack{212113123221\\087968232278}$ | log. |

Dividing the place-row, in the first example, by 8, we obtain $\substack{2131211332132\\7585138894566}$ = Go to Baltimore.

Subtracting 70970970970970, from the place-row in the second example, we obtain $\substack{21113332111312\\38526933388967}$ = Flour is falling.

In the third example, $08796 = \log 12245$; $82322 = \log 66562$; $\log 78 = 6$. Then $\substack{21211312322\\12245665626}$ = Buy corn and rye.

These illustrations, I think, are sufficient to show that a very simple arithmetical process may effectually conceal the meaning of a message from every one but the persons who hold the key to the cypher. The obscurity and consequent security may be increased

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* The first letter may be either L, J, or T.

† The index of the logarithm, is disregarded.

to any desired extent, by adopting processes that are more intricate. But the simpler the cypher, provided it is effectual, the better, and I know of no simpler methods, that merit the name of holocryptic, than those I have just indicated.

## MOMENT OF INERTIA.

BY GERARDUS B. DOCHARTY, LL. D.
Professor of Mathematics in the Free Academy, New York City.

WHEN we arrive at the conclusion, that the *moment of Inertia* is the sum of all the elements of revolving matter into the square of their distance from the axis of revolution, all the after difficulties resolve themselves into analytical ones; and these can only refer to the limits of integration.

The following article is offered, not as having any merit beyond what may be found in the works of Prof. BARTLETT and others who have discussed this subject; but simply for the purpose of assisting those to interpret their symbols, who are pursuing the study of mathematics without the aid of a teacher.

1st. *For the cylinder round an axis perpendicular to its own.*

Let $2\,a =$ its length, $r =$ radius of its base, then

$$k^2 M = \int r^2\, d\,M = \int\!\!\int\!\!\int (x^2 + z^2)\, d\,x\, d\,y\, d\,z$$

$$= \int_{-r}^{r} d\,z \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} d\,y \int_{-a}^{a} d\,x\,(x^2 + z^2)$$

(1) $$= 2\,a \int_{-r}^{r} d\,z \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} d\,y\,(\tfrac{1}{3}\,a^2 + z^2)$$

(2) $$= 4\,a \int_{-r}^{r} d\,z\,(\tfrac{1}{3}\,a^2 + z^2)\sqrt{r^2 - z^2}$$

Let $z = r \sin \varphi$. Then

$$k^2 M = 2\,a\,r^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\,\varphi (\tfrac{1}{3}\,a^2 + \tfrac{1}{4}\,r^2 + \tfrac{1}{3}\,a^2 \cos 2\,\varphi - \tfrac{1}{4}\,r^2 \cos 4\,\varphi)$$

$$= 2\,a\,r^2 \pi\,(\tfrac{1}{3}\,a^2 + \tfrac{1}{4}\,r^2) = M\,(\tfrac{1}{3}\,a^2 + \tfrac{1}{4}\,r^2)$$

$$\therefore k^2 = \tfrac{1}{3}\,a^2 + \tfrac{1}{4}\,r^2$$

The first integral, $2a(\frac{1}{3}a^2 + z^2)$, is the moment of any elementary line of matter within the cylinder in the direction of the axis of the cylinder.

The second integral, $4a(\frac{1}{3}a^2 + z^2)\sqrt{r^2 - z^2}$, is the moment of any elementary plane within the cylinder, parallel to the plane of $xy$. That is, it is the sum of the moments of all the previous elementary lines which have the same $z$.

If we had put $y = r\sin\varphi$, $z = r\cos\varphi$, then $dM = r\,dr\,dx\,d\varphi$. This elementary volume is not the same as the previous one.

$$\begin{aligned}
k^2 M &= \int\!\!\int\!\!\int (x^2 + r^2\cos^2\varphi)\, r\,dr\,dx\,d\varphi \\
&= \int_0^{2\pi} d\varphi \int_0^r r\,dr \int_{-a}^{a} (x^2 + r^2\cos^2\varphi)\,dx \\
&= 2a\int_0^{2\pi} d\varphi \int_0^r r\,dr\,(\tfrac{1}{3}a^2 + r^2\cos^2\varphi) \\
&= a r^2 \int_0^{2\pi} d\varphi\,(\tfrac{1}{3}a^2 + \tfrac{1}{4}r^2 + \tfrac{1}{4}r^2\cos 2\varphi) \\
&= 2ar^2\pi\,(\tfrac{1}{3}a^2 + \tfrac{1}{4}r^2) = M(\tfrac{1}{3}a^2 + \tfrac{1}{4}r^2)
\end{aligned}$$

$\therefore k^2 = \frac{1}{3}a^2 + \frac{1}{4}r^2$ as before.

The first integral is the same as the first above. The second, $ar^2(\frac{1}{3}a^2 + \frac{1}{4}r^2 + \frac{1}{4}r^2\cos 2\varphi)$, is the moment of an elementary plane drawn from the axis of the cylinder to the circumference. The variation of $\varphi$ from 0 to $2\pi$ in the next integral produces the sum of the moments of the whole mass.

The integrations are the same for all homogeneous bodies, differing only in the limits of integration.

2d. *If the bar had been a square prism,* instead of a cylinder, *about an axis parallel to one of the lateral faces and equidistant from them.*

Let $2b$ = the length of one side of the base. Then

$$\begin{aligned}
k^2 M &= \int r^2\,dM = \int\!\!\int\!\!\int (x^2 + z^2)\,dx\,dy\,dz \\
&= \int_{-b}^{b} dz \int_{-b}^{b} dy \int_{-a}^{a} dx\,(x^2 + z^2) = 2a\int_{-b}^{b} dz \int_{-b}^{b} dy\,(\tfrac{1}{3}a^2 + z^2) \\
&= 4ab\int_{-b}^{b} dz\,(\tfrac{1}{3}a^2 + z^2) = \tfrac{8}{3}ab^2(a^2 + b^2) = \tfrac{1}{3}M(a^2 + b^2)
\end{aligned}$$

$\therefore k^2 = \frac{1}{3}(a^2 + b^2)$.

3d. *But, if the axis of motion pass diagonally through the mid-section,* it will be

$$k^2 M = \int_{-b\sqrt{2}}^{b\sqrt{2}} dz \int_{-b\sqrt{2}+z}^{b\sqrt{2}-z} dy \int_{-a}^{a} dx\,(x^2 + z^2)$$

$$= 2a \int_{-b\sqrt{2}}^{b\sqrt{2}} dz \int_{-b\sqrt{2}+z}^{b\sqrt{2}-z} dy\,(\tfrac{1}{3}a^2 + z^2)$$

$$= 4a \int_{-b\sqrt{2}}^{b\sqrt{2}} dz\,(b\sqrt{2} - z)\,(\tfrac{1}{3}a^2 + z^2)$$

$$= 4a \int_{-b\sqrt{2}}^{b\sqrt{2}} dz\,(\tfrac{1}{3}a^2 b\sqrt{2} - \tfrac{1}{3}a^2 z + b\sqrt{2}.z^2 - z^3)$$

$$= \tfrac{16}{3} a b^2 (a^2 + 2b^2) = \tfrac{2}{3} M (a^2 + 2b^2)$$

$$\therefore k^2 = \tfrac{2}{3}(a^2 + 2b^2).$$

The final result *must be the same,* whatever be the *order* of the integrations; but the *difficulty* will not be the same, in general, nor will the meaning of the operations.

For example, in the second set of integrations, if we order them thus: —

$$k^2 M = \int_{-a}^{a} dx \int_{0}^{2\pi} d\varphi \int_{0}^{r} r\,dr\,(x^2 + r^2\cos^2\varphi)$$

$$= \tfrac{1}{4} r^2 \int_{-a}^{a} dx \int_{0}^{2\pi} d\varphi\,(2x^2 + r^2\cos^2\varphi) = \tfrac{1}{4} r^2 \pi \int_{-a}^{a} dx\,(4x^2 + r^2)$$

$$= \tfrac{1}{2} r^2 a \pi\,(\tfrac{4}{3}a^2 + r^2) = M(\tfrac{1}{3}a^2 + \tfrac{1}{4}r^2) \therefore k^2 = \tfrac{1}{3}a^2 + \tfrac{1}{4}r^2.$$

The first integral, $\tfrac{1}{4}r^2(2x^2 + r^2\cos^2\varphi)$, is the moment of a line, drawn perpendicular from the axis of the cylinder to the circumference.

The second integral, $\tfrac{1}{4}r^2\pi(4x^2 + r^2)$, is the moment of any circular section perpendicular to the axis of the cylinder.

4th. *For the cylinder revolving on its own axis.*

$$k^2 M = \int\int\int (y^2 + z^2)\,dx\,dy\,dz = \int\int\int r^2\,dx\,r\,dr\,d\varphi$$

$$= \int_0^r r^3\,dr \int_0^{2\pi} d\varphi \int_{-a}^{a} dx = 2a \int_0^r r^3\,dr \int_0^{2\pi} d\varphi$$

$$= 4a\pi \int_0^r r^3\,dr = a r^4 \pi = \tfrac{1}{2} r^2 M \therefore k^2 = \tfrac{1}{2} r^2.$$

5th. *For the cone revolving on its own axis.*

$$k^2 M = \int\int\int (y^2 + z^2)\, d\,x\, d\,y\, d\,z = \int\int\int r^2\, d\,x\, r\, d\,r\, d\,\varphi$$
$$= \int_0^{b\cos\vartheta} d\,x \int_0^{2\pi} d\,\varphi \int_0^{x\tan\vartheta} r^3\, d\,r = \tfrac{1}{4}\tan^4\vartheta \int_0^{b\cos\vartheta} x^4\, d\,x \int_0^{2\pi} d\,\varphi$$
$$= \tfrac{1}{2}\,\pi \tan^4\vartheta \int_0^{b\cos\vartheta} x^4\, d\,x = \tfrac{1}{10}\,\pi \tan^4\vartheta\, b^5 \cos^5\vartheta$$
$$= \tfrac{1}{10}\,\pi\, b^5 \sin^4\vartheta \cos\vartheta = \tfrac{3}{10}\, b^2 \sin^2\vartheta\,.\,M$$
$$\therefore k^2 = \tfrac{3}{10}\, b^2 \sin^2\vartheta = \tfrac{3}{10}\, r^2.$$

Where $b$ is the slant height of the cone; $\vartheta$ the angle which $b$ makes with the axis; $r$ the radius of the base of the cone, and $k$ the altitude.

The first integral gives the moment of a radius drawn perpendicular from the axis to the surface of the cone, at the distance $x$ from the vertex.

The second, that of a circular section at that distance. If we say

$$k^2 M = \int\int\int r^3\, d\,x\, d\,r\, d\,\varphi = \int_0^r r^3\, d\,r \int_0^{2\pi} d\,\varphi \int_{r\cot\vartheta}^{b\cos\vartheta} d\,x$$
$$= \int_0^{b\sin\vartheta} r^3\, d\,r \int_0^{2\pi} d\,\varphi\, (b\cos\vartheta - r\cot\vartheta)$$
$$= 2\,\pi \int_0^{b\sin\vartheta} r^3\, d\,r\, (b\cos\vartheta - r\cot\vartheta)$$
$$= 2\,\pi\, (\tfrac{1}{4}\, b^5 \sin^4\vartheta \cos\vartheta - \tfrac{1}{5}\, b \sin^5\vartheta \cot\vartheta)$$
$$= \tfrac{1}{10}\,\pi\, b^5 \sin^4\vartheta \cos\vartheta = \tfrac{3}{10}\, b^2 \sin^2\vartheta\,.\,M$$
$$\therefore k^2 = \tfrac{3}{10}\, b^2 \sin^2\vartheta = \tfrac{3}{10}\, r^2 \text{ as before.}$$

The first integral gives the moment of a line (of elementary particles) parallel to the axis of $x$, embraced between the curvilinear surface of the cone and its base.

The second, the moment of all such lines which have the same $r$, or that of a cylindrical surface of elementary particles at any distance $r$ from the axis.

The third, the sum of the moments of *all* such, or that of the whole cone.

If the student shall carefully make himself master of the foregoing examples, he will find no difficulty in determining the moment of inertia of any other homogeneous body of regular figure.

## THE THEOREM OF PAPPUS.

By J. B. Henck, Civil Engineer, Boston.

The famous theorem for measuring surfaces and solids of revolution, so often ascribed to Guldin, may be expressed in two parts thus:

*The surface of revolution described by a line lying wholly on one side of the axis, and in the same plane with it, is measured by the product of the revolving line into the length of the path described by its centre of gravity.*

*The solid of revolution described by a surface lying wholly on one side of the axis, and in the same plane with it, is measured by the product of the revolving surface into the length of the path described by its centre of gravity.*

The claims of Guldin to this theorem have been repeatedly discussed, and the fact made manifest, that the theorem was in existence more than 1200 years before the time of Guldin. Pappus of Alexandria had anticipated the learned Jesuit. Yet many authors, in treating of this theorem, do not even mention Pappus, but either simply call it the "Theorem of Guldin," or expressly ascribe the authorship to him. It may not, therefore, be amiss to present the case more fully, with the citation in the original, of such documents as are of importance, so that mathematicians may decide, whether, in justice to one of the four great geometers of antiquity, the theorem should not be referred to invariably as the *Theorem of* Pappus.

Pappus of Alexandria flourished, according to the common account, about A. D. 380. His principal work is his "Mathematical

Collections." In the Preface to the Seventh Book, after giving an account of the works of the ancient geometers, and of some problems that engaged their attention without being solved, he brings forward the theorem in question, as a proof that his own efforts to contribute something useful have not been in vain. The theorem itself, and a remark appended, are given in these words:*

*Ὁ μὲν τῶν τελείων ἀμφοιστικῶν λόγος συνῆπται ἔκτε τῶν ἀμφοισμάτων καὶ τῶν ἐπὶ τοὺς ἄξονας ὁμοίως κατηγμένων εὐθειῶν ἀπὸ τῶν ἐν αὐτοῖς κεντροβαρικῶν σημείων. Ὁ δὲ τῶν ἀτελῶν ἔκτε τῶν ἀμφοισμάτων καὶ τῶν περιφερειῶν ὅσας ἐποίησε τὰ ἐν αὐτοῖς κεντροβαρικὰ σημεῖα· ὁ δὲ τούτων περιφερειῶν δῆλον ὡς ἔκτε τῶν κατηγμένων καὶ ὧν περιέχουσιν αἱ τούτων ἄκραι, εἰ καὶ εἶεν πρὸς τοῖς ἄξοσιν ἀμφοιστικῶν, γωνιῶν.*

*Περιέχουσι δὲ αὗται αἱ προτάσεις, σχεδὸν οὖσαι μία, πλεῖστα ὅσα καὶ παντοῖα θεωρήματα γραμμῶν τε καὶ ἐπιφανειῶν καὶ στερεῶν, πάνθ' ἅμα καὶ μιᾷ δείξει· καὶ τὰ μὴ προδεδειγμένα καὶ τὰ ἤδη, ὡς καὶ τὰ ἐν τῷ δωδεκάτῳ τῶν Στοιχείων.*

These passages may be translated as follows:

"*The ratio of magnitudes described by a complete revolution* † *is compounded of the ratio of the revolving figures, and of the ratio of the straight lines similarly drawn from their centres of gravity to the axes. The ratio of magnitudes described by an incomplete revolution is compounded of the ratio*

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* From the original of this preface given by HALLEY in his edition of Apollonius de Sectione Rationis. Oxon. 1706. The Greek text of PAPPUS, with the exception of some detached portions, such as that just cited, has never been printed. The world has long expected from the University of Oxford an edition of PAPPUS similar to its editions of Euclid, Apollonius, and Archimedes.

† The words *ἀμφοιστικῶν* and *ἀμφοισμάτων* of the original are unusual forms, and are not given in the lexicons to which I have access. They are doubtless derived from *ἀμφί* and the root *ΟΙΩ, οἴσω*, which supplies several of the forms of *φέρω*; and the meaning of the first appears to be "the effect produced by carrying something around," that is, a solid or surface of revolution, and the meaning of the second, "the thing carried around," that is, the generating area or line. The words *εἰ καὶ εἶεν* present some difficulty. They seem to imply that the angles spoken of might be somewhere else than at the axes, which does not appear possible. Still the phrase may be equivalent to "*which are*," and this is the view taken in our translation.

*of the revolving figures, and of the ratio of the arcs described by their centres of gravity; and the ratio of these arcs is evidently compounded of the ratio of the lines drawn to the axes, and of the ratio of the angles contained by the extremities of these lines at the axes of the magnitudes described."*

He adds: "These propositions, which are almost the same, embrace many and various theorems of lines, surfaces, and solids, all with one demonstration. Of these theorems some have not been before demonstrated, while others have been, such as those in the twelfth book of the Elements."

This enunciation of the theorem differs from the modern one only in this, that, instead of giving directly the measure of a solid or surface of revolution, PAPPUS gives it by giving the *ratio* to each other of any two of these solids or surfaces. From the remark added, he appears to have applied the theorem in various ways, including the measurement of the cone, cylinder, and sphere, which are the solids of revolution of the twelfth book of the Elements of Euclid, to which he doubtless refers.

We of course lose sight of this theorem in the darkness that soon obscured all science. A new day came at last, brightened almost at once by the genius of VIETA, KEPLER, GALILEO, DES CARTES, and CAVALIERI. Not far from this time COMMANDINUS produced his translations and commentaries, and gave a new impulse to the study of the ancient geometers. Among these versions was that of PAPPUS, published in 1588,* thirteen years after the death of COMMANDINUS. His translation of our theorem, as it stands in the reprint of 1660, is thus expressed:

*Perfectorum utrorumque ordinum proportio composita est ex proportione amphismatum, et rectarum linearum similiter ad axes ductarum a punctis,*

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* Pappi Alexandrini Mathematicæ Collectiones a Federico Commandino in Latinum conversæ et Commentariis illustratæ. Fol. Pisauri, 1588; item Bononiæ, 1660.

*quæ in ipsis gravitatis centra sunt. Imperfectorum autem proportio composita est ex proportione amphismatum, et circumferentiarum a punctis, quæ in ipsis sunt centra gravitatis, factarum. Harum circumferentiarum proportio dividitur in proportionem ductarum linearum, et earum, quas continent ipsarum extrema ad axes angulorum.**

In 1615 Kepler published his Nova Stereometria.† In this he introduces the method of generating solids by the rotation of figures about an axis. Of these solids he describes a great number, and gives them various fanciful names, as the olive, apple, citron, &c. Among these he has circular and elliptic rings, the measure of which he obtains by multiplying the area of the rotating circle or ellipse by the length of the circumference described by their centres. He does not, however, use the centre of gravity, as such, in any of his measures.

We come now to Guldin, who was born in 1577, at St. Gall in Switzerland, and named Habakkuk. At the age of twenty he renounced Protestantism, and entered the Society of Jesuits as a "temporal coadjutor," changing his name to Paul. He was sent to Rome by his superiors to study mathematics; and he afterwards taught that science in the colleges of his order at Rome, Grätz, and Vienna. His principal work is his Treatise, in four books, on the Centre of Gravity.‡ He died in 1643.

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* In this version the words "*utrorumque ordinum*" answer to ἀμφοιστικῶν, where it first occurs in the Greek text of Halley, and a blank supplies its place near the end; while ἀμφοισμάτων is simply changed to *amphismatum*. Commandinus certainly enlightens us very little as to the meaning of these words. Towards the close, "*earum quas*" seems to be a misprint for *eorum quos*, referring to *angulorum*.

† Nova Stereometria Doliorum vinariorum, etc. Accessit Stereometriæ Archimedeæ Supplementum. Lincii, 1615.

‡ Pauli Guldini Sancto-Gallensis, e Societate Jesu, de Centro Gravitatis trium specierum quantitatis continuæ. Liber Primus. Fol. Viennæ Austriæ, 1635; Liber Secundus, 1640; Liber Tertius, 1641; Liber Quartus, 1641.

GULDIN appears to have been a close student of PAPPUS, whom he cites, in the course of his book, about a dozen times. In several instances he quotes sentences, and these he gives in the very words of the version of COMMANDINUS. In his second book, after considerable heralding, he introduces as his own discovery the following Rule:

*Quantitas rotanda in viam rotationis ducta, producit potestatem rotundam uno gradu altiorem, potestate sive quantitate rotata.*

*"The quantity to be rotated, multiplied by the path of rotation, produces a circular power one degree higher than the power or quantity rotated."*

In explanation of this rule GULDIN observes, that there are three geometrical powers; namely, the line, a power of the first degree, produced by the motion of a point; the surface, a power of the second degree, produced by the motion of a line; and the solid, a power of the third degree, produced by the motion of a surface. These powers are "circular powers," when the generating motion is circular. The "path of rotation" is the path described by the centre of gravity. His third Corollary is:

*Si autem tam quantitates rotandæ, quam viæ sive radii rotationis sint inæquales, sequitur ulterius potestatum proportionem esse compositam ex ratione quantitatis rotatæ unius, ad quantitatem rotatam alterius; et ex ratione viæ vel radii illius unius, ad viam vel radium hujus alterius.*

*"But if both the quantities to be rotated and the paths or radii of rotation are unequal, it follows, further, that the ratio of the powers is compounded of the ratio of the rotated quantity of the one to the rotated quantity of the other, and of the ratio of the path or radius of the former to the path or radius of the latter."*

This is of course a reproduction of the theorem of PAPPUS, and the third Corollary presents it in almost the same form. It would ·rhaps, be just to accuse GULDIN of deliberate plagiarism; for,

though he probably read the passage cited above from the version of COMMANDINUS, he might possibly fail to get a correct idea of the theorem from this faulty translation. Still the ingenuity to do so would seem to be much less than that required to enable him to discover it independently. His attempts at originality on other subjects are not so successful as to give us a very high opinion of his inventive powers; while the general tone of his work, his evident conceit, self-satisfaction, and boastfulness, do not predispose the reader to admit his claims in the present instance without some scrutiny.

In regard to a demonstration of the theorem, we do not know that PAPPUS ever gave one. A preface was not the place for a demonstration, and if he resumed the subject elsewhere, we do not know it. GULDIN, however, seems to have attempted a demonstration, which amounts to this: The quantity generated is equal to the revolving figure multiplied by the length of some one of the circumferences described by its points. The circumferences described by the points nearest the axis are too small for this purpose, and those described by the points farthest from the axis are too large. Therefore, a circumference must be found at such a distance from the axis that the excess outside of this distance just balances the deficiency within it, and this line must be one that is unique (*unica*). All this will be done, if we take for that circumference the path described by the centre of gravity, which is a unique line (*quæ est sola et unica*). He does not, however, show how the balancing of which he speaks takes place, which is the essential point, but contents himself with asserting that such is the fact. He adds, that this thing requires no other demonstration; but it will be sufficient to show its truth by induction; that is, by ascertaining that the results by his rule agree precisely, in numerous examples, with those obtained otherwise.

The first real demonstration appears to have been given in a

work by CAVALIERI.* GULDIN, in his fourth book, attacks the Method of Indivisibles of CAVALIERI, accusing him of departing from the rigor of the ancient geometry. CAVALIERI, according to MONTUCLA, retorts, that GULDIN's demonstration, just alluded to, displays any thing but the ancient rigor, and says, that one of his pupils, ANTONIO ROCCHA, furnished him with a demonstration of the theorem, long before GULDIN published his work. This demonstration he gives.

It may be interesting, before closing, to give briefly some of the opinions of writers who have touched upon the claims of PAPPUS and GULDIN.

LEIBNITZ,† speaking of this theorem, and giving an extension of it, remarks: "PAPPUS had briefly indicated what GULDIN showed more fully." (*Pappus subindicaverat quod Guldinus expressius ostendit.*)

HALLEY ‡ says, that one of his principal motives in giving the Greek text of the preface to the Seventh Book of PAPPUS was, "that it might become plain to all, that the Centrobaric Method of GULDIN was known to the ancients also." (*Ut palam fiat omnibus Regulam Guldini Centrobaricam . . . . . . ipsis etiam Veteribus innotuisse.*)

VARIGNON,§ in a memoir in which he demonstrates the theorem itself and several extensions of it, does not even allude to PAPPUS, but ascribes the discovery to GULDIN.

MONTUCLA || discusses the whole question fully, and then asks: "After these developments, ought we not to give to this fertile prin-

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* Exercitationes Geometricæ Sex. Bononiæ, 1647. This work, and several others of the same age, which would more fully illustrate the views of Guldin's immediate successors, are unfortunately not to be found in the libraries of this vicinity.

† Acta Eruditorum, 1695, p. 493.

‡ Preface to Apollonius de Sectione Rationis. Oxon. 1706.

§ Mémoires de l'Académie. 1714, p. 77.

|| Histoire des Mathématiques, Vol. I. p. 329.

ciple the name of PAPPUS, instead of that of GULDIN, who merely reproduced it, whether he did it by his own ingenuity, or whether this passage of PAPPUS gave him the idea?" He adds, after mentioning GULDIN's evident acquaintance with PAPPUS: "Still I do not care to accuse GULDIN of plagiarism; although it seems to me difficult to free him from the charge."

BOSSUT,* speaking of PAPPUS, says: "I ought to add, in praise of PAPPUS, that we find at the end of the Preface to his Seventh Book a quite distinct idea (*une idée assez distincte*) of the famous theorem commonly attributed to the Jesuit GULDIN."

FRISI,† after demonstrating the theorem, remarks: "This is the celebrated theorem which some contend was indicated by PAPPUS ALEXANDRINUS, but which was really first given for circular and elliptic rings by KEPLER, and afterwards proposed generally by GULDIN, and proved by a sort of induction, and which, finally, was demonstrated by ANTONIO ROCCA of Piacenza, by means of the principles of our CAVALIERI, in nearly the same way that we have here followed."

WHEWELL ‡ says: "A property of the center of gravity, discovered by PAPPUS, and republished about 1640 by GULDIN, has since borne the name of its more recent enunciator."

And, lastly, CHASLES § says: "The famous theorem of GULDIN . . . . . . is found in the Mathematical Collections, and appears to have been the discovery of PAPPUS himself." And afterwards, speaking of GULDIN, he adds: "This rule was unnoticed, when GULDIN dis-

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* Histoire des Mathématiques, Vol. I. p. 47.

† Paulli Frisii Opera, 3 Vols. 4to. Mediolani, 1782–85, Vol. II. p. 30.

‡ Mechanics. Cambridge, 1819, p. 111.

§ Aperçu Historique sur l'Origine et le Développement des Méthodes en Géométrie, Bruxelles, 1837, p. 29 et 57.

covered it in his turn, and used it to solve problems that were difficult and intractable by other methods."

All these writers recognize the claims of PAPPUS, except VARIGNON, who does not mention them, and FRISI, who seems inclined to ignore them. Among writers not yet cited, PRONY, BORGNIS, BOUCHARLAT, YOUNG, MOSELEY, DUHAMEL, and WEISBACH do not mention PAPPUS in connection with this theorem, but refer to it as GULDIN'S Theorem; FRANCŒUR and O. GREGORY ascribe its discovery to PAPPUS; while WALTON * and PRICE † refer to it as the Theorem of PAPPUS.

The evidence, so far as we have been able to collect it, is now before the reader. Let him judge for himself. It is a beautiful custom, nowhere more fully recognized than among mathematicians, to connect with a new truth the name of its author; and if in this case the wrong name has received the honor, it is not too late to correct the error. We have only to reverse the prevalent practice; so that, instead of mentioning PAPPUS incidentally or not at all, and quoting the theorem as GULDIN'S, we shall, if occasion serve, mention the *applications* made by GULDIN, but the theorem itself we shall quote as the THEOREM OF PAPPUS.

## ON THE COURSING JOINT CURVE OF AN OBLIQUE ARCH IN THE FRENCH SYSTEM.

BY DEVOLSON WOOD, C. E.
Assistant Prof. of Engineering in the Univ. of Michigan.

1. IN constructing the development of the arch the draughtsman must either construct the coursing joints geometrically or else

* Problems in Theoretical Mechanics.

† Infinitesimal Calculus, 3 Vols. 8vo. 1852–1856.

construct them from the equation of the curve. I propose to find the equation of the curve when the face is semicircular.

Suppose the arch is intersected by an infinite number of planes parallel to the face, they will cut an infinite number of ring joints. In the French system the coursing joints of either the intrados or extrados intersect these joints at right angles.

Let $B'$ be the origin, and $B'E$ be the axis of $x$ which is the springing line; $r = B'C =$ radius; $r\varphi =$ the length of arc from the springing line to the curve $= ad$; $r\varphi_1 =$ initial arc $= B'D'$; $i =$ inclination of the curve to an element; $\theta = ACB' - 90° =$ skew angle.

2. To find $i$ in terms of $\varphi$ and $\theta$. With $C$ as a centre and radius equal to unity describe the arcs $AB$, $AD$, and $DB$. We shall thus form a spherical triangle $ADB$, in which $AB = 90° + \theta$, $DB = \varphi$, $ADB = 90° + i$, $ABD = 90°$. Hence, by Napier's rules,

$$\sin\varphi = \cot(90° + i)\tan(90° + \theta),$$

or

$$\sin\varphi = \tan i \cot\theta,$$

$$\therefore\ \tan i = \sin\varphi\tan\theta.$$

This value of $i$ is the same for any value of the radius. In the differential triangle $abc$ we have

$$\frac{bc}{ab} = \frac{r\,d\varphi}{dx} = \tan i = \sin\varphi\tan\theta.$$

Hence

$$\int_0^x dx = r\cot\theta\int_{\varphi_1}^{\varphi}\frac{d\varphi}{\sin\varphi};$$

or

$$x = r\cot\theta\,(\log\tan\tfrac{1}{2}\varphi - \log\tan\tfrac{1}{2}\varphi_1),$$

from which the curve in the development may be constructed.

In a German work on bridges, the following formula is given, but without investigation; namely,

$$x = \frac{r}{\sin\theta} \log \cot \tfrac{1}{2} \varphi,$$

which is evidently incomplete, since $\varphi$ can have but one value for any assumed value of $x$; whereas it should have several values, one for each coursing joint which depends upon the initial value $\varphi_1$ of $\varphi$ for $x = 0$.

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## THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 148.]

### SECTION II.

#### ON THE MOTIONS AND FIGURE OF A FLUID SURROUNDING THE EARTH.

11. The results obtained in this and the following section, will be upon the hypothesis that the motions of the fluid are not resisted by the earth's surface. The motions and figure of the fluid must be such as to satisfy equations (13), and also the condition of continuity. To determine them in the general case in which both $a$ and $g$ are functions of $h$, $\theta'$ and $\varphi$, would be very difficult. We shall take here the special case only in which $a$ is a function of $\theta'$ which increases or decreases, from the equator to the pole, and in which $g$ is regarded as constant at all parts of the earth's surface, and throughout the whole range of altitude. Equations (13) become in this case, when the fluid is elastic,

$$(20)\quad \begin{aligned} g\, D_{\theta'} h &= r^2 \sin\theta \cos\theta\,(2\,n + D_t \varphi)\, D_t \varphi - r^2 D_t^2 \theta - g\, h\, D_{\theta'} \log a \\ g\, D_{\varphi} h &= -\,2\, r^2 \sin\theta \cos\theta\,(n + D_t \varphi)\, D_t \theta - r^2 \sin^2\theta\, D_t^2 \varphi. \end{aligned}$$

In inelastic fluids we have $\log k$ instead of $\log a$.

12. To determine completely the motions and figure of the fluid which, would satisfy the conditions for any initial state of the fluid, upon the hypothesis that the motions are entirely free from resistances arising from the motions of the particles amongst themselves, would be impossible. But since in all fluids there are slight resistances to the motions of the particles amongst themselves, which, however small, eventually destroy all oscillatory or wave motions depending upon the initial state of the fluid, and reduce the motions of the particles amongst themselves to the minimum which satisfies the conditions, it is not necessary to integrate the equations generally, but merely to satisfy them with the least possible motion of the particles amongst themselves. Hence both the motions and density of the fluid at any place, and likewise its figure, must be independent of the time, and therefore constant.

13. Since $D_\phi h$ in the last of equations (20) can only have a value arising from an oscillatory or wave motion of the fluid, which would soon be destroyed, it must be put equal 0, and then the equation gives by integration for each particle supposed to be entirely free from the resistances arising from the motions of the particles amongst themselves,

$$r^2 \sin^2 \theta \, (n + D_t \varphi) = c, \tag{21}$$

in which $c$ is a constant depending upon the initial motion of the particle. Let

$R$ be the radius of the earth regarded as constant,
$m$ the mass of the fluid, and
$l$ its uniform depth when at rest relative to the earth.

As the quantity of motion in the whole mass cannot be affected by the mutual actions of the particles upon each other, we have, even in the case in which the particles are not free from mutual resistances,

$$\int_m r^2 \sin^2 \theta \, (n + D_t \varphi) = \int_m c = Cm, \tag{22}$$

in which $C$ is a constant depending upon the initial motions of all the particles.

The first member of this equation expresses the sum of the areas projected upon the plane of the equator, arising from the absolute motions of all the particles for a unit of time, and hence this sum is constant.

14. If we put $v$ for $D_t\varphi$ belonging to the initial state of the fluid, the last equation gives, neglecting quantities of the order of the range of altitude compared with the earth's radius,

$$Cm = R^2 \int_m \sin^2\theta\,(n+v),$$

$$= R^4 \int_0^l {}_N \int_0^{2\pi} {}_\phi \int_0^\pi {}_\theta\, k \sin^3\theta\,(n+v),$$

$$= \tfrac{2}{3} R^2 m\,(n+v')$$

in which

$$v' = \frac{3R^2}{2m} \int_0^l {}_N \int_0^{2\pi} {}_\phi \int_0^\pi {}_\theta\, k\,v \sin^3\theta.$$

Hence,

$$C = \tfrac{2}{3} R^2 (n+v').$$

In the preceding integration $k$ is supposed to be independent of $\theta$. If the density should vary considerably with the latitude, it would affect the preceding result slightly.

When, by the mutual actions of the different strata upon one another, $D_t\varphi$ becomes the same at all altitudes, upon the same parallel of latitude, $c$ then becomes equal to $C$, and equation (21) gives

$$(23)\qquad D_t\varphi = \frac{C}{R^2 \sin^2\theta} - n = \frac{2\,(n+v')}{3\sin^2\theta} - n.$$

This value of $D_t\varphi$ satisfies the last of equations (20), and since it gives a uniform motion of all the particles of the fluid upon the same parallels of latitude, as much fluid flows from any place as flows into it, while the density, from what has been stated, remains the same, and hence it also satisfies the condition of continuity.

15. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, equation (23) becomes

$$(24) \qquad D_t \varphi = \left(\frac{2}{3 \sin^2 \theta} - 1\right) n,$$

substituting this value of $D_t \varphi$ in the first of equations (20) we get, by putting $R$ for $r$,

$$(25) \quad g D_\theta' h = R^2 n^2 \sin \theta \cos \theta \left(\frac{4}{9 \sin^4 \theta} - 1\right) - R^2 D_t^2 \theta - g h D_\theta' \log \alpha.$$

Since the last term of this equation is a function of $h$, the height of the strata, the equation can only be satisfied by counter currents of the strata between the equator and the poles; and as we have seen that the figure of the fluid, and its density at the same place, are constant, in order to satisfy the condition of continuity, these currents must be such as to satisfy, for every vertical column of the fluid, the following equation,

$$(26) \qquad \int_m D_t \theta' = 0.$$

In order to satisfy this condition, $h$, which is a general integral, must have a negative constant added to it. Hence at a certain altitude the last term of (20) vanishes, and the fluid there has no motion towards or from the equator.

If the density increases towards the poles, this term is positive for the lower strata, but negative for the upper ones, and hence the motion is toward the equator below, and from it above. If the density decreases towards the poles, the motions are the reverse.

If there were no resistances of any kind, the motions would be continually accelerated so long as the density is different between the equator and the poles; but where there are slight resistances, the motions are only accelerated until the resistances become equal to the accelerating force.

16. Since the last two terms of (25) have a very little effect upon the value of $h$, and consequently upon the figure of the fluid,

in comparison with the remaining term of that member, unless the difference of density, and the motion of the fluid between the equator and the poles, are very great, we shall neglect them, and determine the figure depending upon the remaining term arising from the earth's rotation. Equation (25) gives by integration in this case, since $D'_\theta h$ does not differ sensibly from $D_\theta h$,

$$2gh = -R^2n^2\left(\frac{4}{9\sin^2\theta} + \sin^2\theta\right) + C.$$

If $h'$ be put for the value of $h$ at the equator, putting $\sin\theta = 1$, we get

$$2gh' = -\tfrac{13}{9}R^2n^2 + C.$$

Hence,

$$h = h' + \frac{R^2n^2}{2g}\left(\tfrac{13}{9} - \frac{4}{9\sin^2\theta} - \sin^2\theta\right). \tag{27}$$

Since one of the terms in this value of $h$ has $\sin\theta$ in the denominator, whatever be the value of $h$ at the equator, it must become 0 towards the poles, and the surface of the fluid meet the surface of the earth; and this must be the case, however large the terms which have been neglected. Hence *the fluid, however deep it may be at the equator, cannot exist near the poles.*

17. If $\theta_0$ be the value of $\theta$ where $h = 0$, the last equation gives

$$\sin^4\theta_0 - \left(\frac{2g}{R^2n^2}h' + \tfrac{13}{9}\right)\sin^2\theta_0 = -\tfrac{4}{9}, \tag{28}$$

which determines $\theta_0$ when $h'$ is given.

If we put $\theta_1$ for the value of $\theta$ where $h$ is a maximum, equation (25), putting $D'_\theta h = 0$, and neglecting the last two terms, gives

$$\sin^2\theta_1 = \tfrac{2}{3}, \tag{29}$$

This gives $\theta_1 = 55°$ nearly, answering to the parallel of 35°, where $h$ is a maximum.

If, therefore, we assume $h'$, equation (27) gives the figure which the fluid assumes, *which must be somewhat as represented in the external*

*part of* Fig. (1), *the surface of the fluid being slightly depressed at the equator, having its maximum height about the parallel of* 35°, *and meeting the surface of the earth towards the poles.*

18. In the applications of the preceding equations we must put

$$R = 3956 \text{ miles} = 20887680 \text{ feet},$$

$$n = \frac{2\pi}{(23 \times 60 + 56)\,60} = .000072924$$

$$g = 32.2 \text{ feet}.$$

Hence $\quad Rn = 1523.2$ feet, and $\dfrac{R^2 n^2}{2g} = 36017$ feet.

With these values, if we assume $h' = 5$ miles, (28) gives $\theta_0 = 28°\ 30'$ for the polar distance of the parallel where the surface of the fluid, or the stratum of equal pressure, meets the surface of the earth.

Fig. 1.

If in (27) we substitute for $\sin\theta$ its special value in (29), we obtain $h - h' = 4002$ feet for the excess of the height of the fluid at its maximum, above its height at the equator; which is a constant independent of the amount or depth of the fluid.

19. If we put $D_t\,\varphi = 0$ in (24), it gives

$$\sin^2\theta = \sin^2\theta_1 = \tfrac{2}{3}. \tag{30}$$

Hence, *the latitude of no motion of the fluid east or west, is the latitude of the maximum of* $h$.

20. If $\sin^2\theta < \frac{2}{3}$ in (24), $D_t\,\varphi$ is positive, but if $\sin^2\theta > \frac{2}{3}$, it is negative. Hence, *between the parallels of* 35° *and the poles, the motion of the fluid is eastward, but between those parallels and the equator it is toward the west.*

21. The lineal velocity of the fluid east or west relative to the earth's surface is $R \sin\theta\, D_t\varphi$. Representing it by $v''$, equation (24) gives

$$v'' = Rn\left(\frac{2}{3\sin\theta} - \sin\theta\right). \tag{31}$$

Putting $\sin\theta = 1$, this equation gives $v'' = -\frac{1}{3}Rn = 508$ feet for the velocity of the fluid westward per second at the equator. Towards the poles it is evident, from an inspection of the preceding equation, that the velocity must become very great.

The east and west motions of the fluid, as well as its figure, are represented by Fig. (1), the different lengths of arrows representing, in some measure, the different velocities of the fluid.

22. The whole lineal velocity of the fluid east, arising from both the earth's rotation and the velocity of the fluid relative to the earth's surface, is $R\sin\theta\,(n + D_t\varphi)$. Representing this by $v'''$, equation (24) gives

$$v''' = \frac{2Rn}{3\sin\theta}. \tag{32}$$

Hence this velocity is inversely as the distance from the axis of rotation, which is a necessary consequence of the preservation of areas, as shown in § 13. For as the fluid in moving from the equator towards the poles approaches the axis of rotation, it must have its velocity increased, and in receding from the axis it must be decreased, just as a planet is accelerated in its perihelion but retarded in its aphelion. The reasoning, therefore, of those who, in attempting to explain the trade winds, assume that the fluid, in moving towards or from the equator, has a tendency to retain the same lineal velocity, is erroneous.

23. If, instead of a state of rest relative to the earth's surface, we suppose that the fluid has an initial angular velocity, we must put $n + v'$ instead of $n$ in the preceding equations.

## ON PRACTICAL GEOMETRICAL METHODS OF LOCI.

BY D. H. MAHAN,
Professor of Engineering in the U. S. Military Academy, West Point, N. Y.

EDITOR OF THE MATHEMATICAL MONTHLY:

SIR, — Permit me to call the attention of your younger mathematical readers to the subject of the practical geometrical methods of *loci*, as affording both an interesting and profitable exercise in the use of mathematical instruments, particularly for those who are pursuing scientific studies with the view of becoming civil engineers or architects. I have selected two cases out of the many that have fallen under my notice in their applications to practice.

Prob. To inscribe within a given rectangle another, one side of which is given.

Let $A B$ be the given rectangle; $a b$ the one to be found, the side $a c$ of which is given.


An examination of the Fig. will show that the diagonals of the two rectangles pass through the same point $O$. From $O$ then, with any assumed radius $O a'$ describe an arc, and from $a'$ set off the chord $a' c' = a c$. If the point $c'$ falls without $B C$, take any radius $O a'' < O a'$ and repeat the same construction, thus determining the point $c''$ within $B C$. Having, in this way, found as many points $c'$, $c''$ &c., as desired, on either side of $B C$ draw through them the line $c' c c''$, &c. This line will be the *locus* of the required conditions, and the point $c$, where it intersects $B C$, will be one angle of the inscribed rectangle to be found, from which the others are readily determined by the inverse order of the construction.

*Remark.* This problem finds an application in HOWE'S Truss for

bridges, &c., in determining the position of the oak or iron steps, or blocks against which the diagonal braces abut.

Prob. Having given a circle, a right line and a point without the circle, to find the tangents to the circle to which if perpendiculars be drawn from the given point, the points of intersection of these lines will be found on the given line.

Let $O$ be the centre of the given circle; $O\,r'$ its radius; $C$ the given point, and $A\,B$ the given right line. Draw a series of tangents as $t'$, $t''$, &c., to the circle, and from $C$ demit upon them the perpendiculars $p', p''$, &c. The points of intersection of these lines will give a curve $c\,c''$, &c., which is the *locus* of the conditions; and where it intersects the line $A\,B$ as at $c$ and $c''$ will be the required points.

*Remarks.* 1st. In examining the data of the Prob. it will be seen that by joining the point $C$ with $O$ the centre of the circle, and on this line, as a diameter describing a circle, and from $C$ drawing any number of chords $C\,x'$, $C\,x''$, &c., and on these setting off the distances $x'\,c'$, $x''\,c''$, &c., each equal to $O\,r'$, the points $c', c''$, &c., will be the points of the *locus* as before. Moreover, the conditions will still be satisfied by prolonging the chords and setting off upon them the distances $x'\,d'$, $x''\,d''$, &c.; this will give a branch of the curve within which the circle and the point $C$ will be found.

2d. An examination of the preceding constructions will show several of the remarkable properties of this curve, which, *in connection with the circle*, presents two symmetrical *gourd-shaped* branches; $r_2\,r'\,c''$ $r_1\,C\,d^n$ being one. The curve will be tangent to the circle at $r_1$ and $r_2$, where the line $C\,O$ cuts the circle; it will pass through the point

$C$, forming there a cusp or knot. The equation of the curve will evidently be of the 4th degree, with equal pairs of roots for the ordinates referred to an axis perpendicular to $O\,C$, at $r_2$; and of unequal roots for the abscissas referred to $r_2 C$ as an axis.

3. The problem may evidently be enunciated thus: Having a given chord in a circle, to draw, from a given point, another chord, such that the portion of it intercepted between the given chord and its arc shall be equal to a given line. This problem, if I recall accurately an investigation of it made by me some years back, reduces, for its solution, to the celebrated puzzle of the tri-section of an angle.

This example is also from a practical application; but is given only for the purpose of illustrating the utility of such simple mechanical means for solving problems, otherwise demanding, for their investigation, a resort to analytical geometry which is not at the disposal of many who might resort to this method.

In connection with this subject, I would call the attention of your young readers to the admirable applications of it by my old friend and professor years ago, *General* PONCELET, *Member of the French Institute;* some one of whom would do the scientific civil engineers of our country a great service by translating his *Memoirs on the Stability of Retaining Walls and Arches,* in which he has used these methods in testing two of the most difficult applications of the higher analysis.

To instructors I will permit myself to add, that, as an introduction to practical applications of geometrical constructions, exercises of this character at the black-board, with a scale, dividers, and a triangle, will be found of the greatest service to the young engineer.

# Mathematical Monthly Notices.

*First Lessons in Geometry.* By THOMAS HILL. *Facts Before Reasoning.* Boston: Hickling, Swan and Brown.

OUR readers already know Mr. Hill's views, as set forth in his article "On the order of Mathematical studies," in relation to the early introduction of the study of Geometry, and its value as a mental discipline; first, by means of forms which address the mind through the senses; second, by means of a simple but logical statement of the coördinated series of truths of Geometry, to educate the powers of conception; and lastly, the rigorous demonstration and application of these truths, to strengthen and develop the reasoning and inventive powers. This little book is intended to aid the child in the second stage of its progress in the study of Geometry; and, so far as we know, this is the only one that does not include reasoning and a knowledge of the symbols and operations of arithmetic, which the child at this age is not supposed to have acquired. This idea of putting Geometry before Arithmetic is not new, for Mr. Hill himself has diligently preached the doctrine for the last twelve or fourteen years; but while others may have implied it, we believe he was the first to give it prominence and vitality by putting it into practice. Nor has Mr. Hill confined himself to the simple elements of Geometry, for the "First Lessons" contain a clear and simple statement of the characteristic properties of the Circle, Ellipse, Parabola, Hyperbola, Catenary, the Cycloidal curves, and Caustics, with their Involutes and Evolutes. This book is used in all the schools of Waltham with marked success; but this might be attributed to the energy and enthusiasm which Mr. Hill has infused into these schools by his personal influence, were it not a fact that all thorough teachers, in many schools in different parts of the country who have given the book and the idea which it represents a fair trial, have been equally successful. But on the other hand we feel sure that many teachers would as signally fail; and simply for want of the proper knowledge of, and interest in, the subject. We believe that if this idea is ever thoroughly and generally incorporated into our system of elementary instruction, it will be mainly through the influence of our Normal Schools; the only institutions in which the theory and practice of teaching are studied in connection, and by those who will, in the discharge of the duties of their profession, practise what they believe to be true in theory, whatever may be their natural tastes or preconceived opinions.

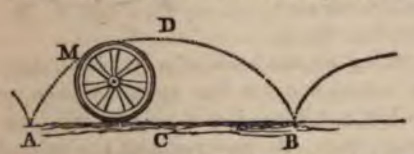
It is generally admitted that one of the greatest, if not the greatest evil at present existing in our primary schools, is the want of proper amusement of an instructive character for small children, while learning the alphabet, and indeed until they have so far advanced as to begin to study for themselves. During this period of unprofitable restlessness and weariness, just the time when pleasant impressions of the school-room, of books, and of study should be received, children often acquire a distaste for these things which they never outgrow; and we believe that hardly another subject can be named so well adapted to amuse and instruct as Geometry taught by means of sets of blocks, and drawing on the slate and blackboard.

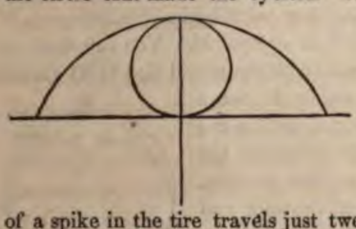
And such a course would be preparatory to the study of the "First Lessons," in connection with which the blocks and drawing would still be used by way of illustrating and simplifying the definitions and descriptions.

The study of Geometry is generally acknowledged as the best means of cultivating the habit of mental attention and concentration; and the desirableness of early disciplining these faculties has led to the wide introduction of the study in all but the lowest grades of schools. Besides, many teachers are agreed that Geometry should precede Algebra, and we find that in many institutions the course of study is arranged accordingly. Mr. Hill proposes to carry the idea still further by making the study of Geometry precede Arithmetic, and to this end has prepared the little work before us. He says "geometrical facts and conceptions are easier to a child than those of arithmetic, but arithmetical reasoning is easier than geometrical." This suggests the order of these studies as a discipline of the reasoning powers. Arithmetical reasoning first, and geometrical afterwards.

We most sincerely commend this little volume and the whole subject to the attention of teachers and all in any way interested in the subject of education, and add the following chapters to show how clear and vivid an idea of even the higher curves can be obtained by description alone, and how decidedly this habit of conceiving of a curve in kind must assist the student afterwards to apply his analysis and compute it in magnitude.

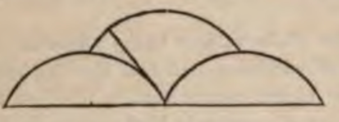
CHAPTER XXVII. *About a Wheel Rolling.* — 1. When a wagon is going upon a straight and level road, look at the head cf a spike in the tire of one of the wheels, and you will see that it moves in beautiful curves, making a row of arches that is called a cycloid. 2. That is to say, a cycloid is the path of a point in the circumference of a circle rolling on a straight line. You can draw part of a cycloid by putting the point of your pencil into a little notch in the edge of a spool, and tying it fast, so that the point of the pencil shall be kept just at the edge of the spool; and then rolling the spool carefully and slowly against the inside of the frame of the slate. 3. You will see, I think, that each arch in the cycloid must be just as high from C to D as the diameter of the circle that makes it; and just as wide at the bottom, from A to B, as the whole circumference of the circle. 4. But you will have to study Geometry a good while, before you can prove the other interesting things which I am going to tell you. You can easily understand what I am going to tell you; but you cannot understand how I know it, as you can what I told you in the last section. 5. The length of the curve, A D B, in each arch of a cycloid, is just four times the height of the arch; that is, four times the diameter of the circle that made the cycloid. 6. The whole space that is enclosed between the arch of the cycloid and the straight line on which it stands, is just three times as large as the circle that made the cycloid. So, when a circle is inscribed between the arch and the line, the curious three-cornered figures on each side of the circle are each exactly as large as the circle itself. 7. Now, if you have studied Arithmetic, you will understand that, if a wheel is three feet in diameter, the head of a spike in the tire travels just twelve feet from where it leaves the ground until it touches the ground again. The spots where it touches the earth will be nine feet and three sevenths of a foot apart. And the space between its path and the ground will be three times eleven fourteenths of nine square feet; that is, twenty-one square feet and three fourteenths of a square foot.



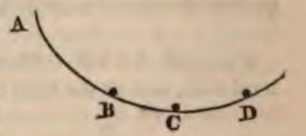



CHAPTER XXVIII. *More about a Rolling Wheel.* — 1. The head of a spike in the tire of a

rolling wheel is moving, at each instant, at right angles to a line joining it to the bottom of the wheel. 2. That is to say, if a straight line is drawn from the bottom of the rolling wheel to the head of the spike, and if a tangent to the cycloid is drawn through the head of the spike, this straight line will be at right angles to this tangent. 3. And this straight line is exactly half of the radius of curvature of that point in the cycloid. So that at the top of an arch of the cycloid the radius of curvature will be twice the diameter of the circle, and as you go down the arch the radius of curvature will be shorter and shorter, until just at the foot of the arch the radius of curvature will be of no length at all. 4. The evolute of a cycloid is a cycloid of exactly the same size. That is to say, if we should fasten a string in the point



between two arches of a cycloid, just long enough to wrap on the curve up to the middle of the arches, its end as it is wrapped and unwrapped, would move in a cycloid exactly like that to which it was fastened. 5. If a cycloid be turned upside down, and we fancy the inside of it to be very exceedingly slippery, then there are two curious things about it. If I want to slide any thing from A down to B, there is no curve, nor straight line, down which a thing would slide so quickly as down the cycloid. If a hill was hollowed out in that shape, sleds would run down it faster than they could down any other shaped hill of the same height and the same breadth at the bottom.




6. The second curious thing about sliding on the inside of a cycloid is, that it takes always exactly the same time to slide to the bottom, however high up or low down you start. If A, in the last figure, is the top of such a hill, and C the lowest point, it will take a sled exactly as long to go from B to C, as to go from A to C. But this, you must remember, is only when we imagine the hill and the runners of the sleds to be, both of them, perfectly slippery; so that there shall be no rubbing. In that case, if the road from A to C was two miles long, it would only take a sled twenty-eight seconds to come down the whole length. And, if it starts from any other place on the road, say from B, it will still take twenty-eight seconds to get to C. 7. If the road from A to C is half a mile long, a sled will come down in fourteen seconds. 8. If a board is sawed out in the form of a cycloid, and a little gutter made on the inside of the curve, you can try this by holding two marbles, say one at A and the other at D, and letting go of them at the same instant. They will meet exactly at C, one coming the whole way A C, while the other is coming the short distance D C.

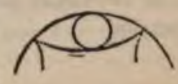
CHAPTER XXIX. *Wheels Rolling Round a Wheel.* — 1. When one circle rolls around another, instead of rolling on a straight line, any point in the circumference of

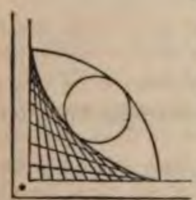
the rolling circle travels in a curve called an epicycloid. You can draw an epicycloid by rolling carefully the spool (with the pencil tied to it) around some round thing held still on your slate. 2. Set a lamp on a table in one corner of the room, and, in the farthest corner of the room, on a table of nearly the same height, set a bright tin cup, or a glass tumbler, nearly full of milk. On the surface of the milk you will see a bright curve shaped like the inner line in this figure. It is an epicycloid; such as would be made by a circle of one quarter the diameter of the cup rolling on a circle half the size of the cup.

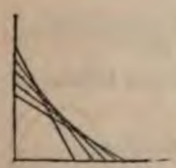

You can make it by daylight, by setting the cup of milk in the sunshine, early in the morning or late in the afternoon. 3. Epicycloids

will be of different shapes, according to the proportion which the two circles bear to each other. The smaller the rolling circle is in proportion to the other, the more nearly will an arch of the epicycloid be like an arch of the cycloid. 4. The evolute of an epicycloid is a smaller epicycloid of the same shape; and the evolute of that evolute must be a still smaller epicycloid. So that we may fancy epicycloids packed one within another like pill-boxes. 5. The epicycloid of section second is sometimes called by children the cow's foot in a cup of milk. The figure in the margin represents this epicycloid with its nest of evolutes packed one within the other. If a string is fastened at the point where the arches of the epicycloid come together, and is just long enough to wrap round to the middle of the arch, then, as it unwraps, the end will move in a larger epicycloid of exactly the same shape. 6. When the circles are of the same size, the epicycloid will have but one arch. The ends of the arch will come together at the same point. The figure in the margin will show the shape of this epicycloid and its evolutes.

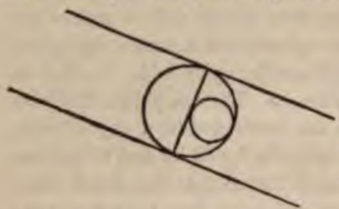
Chapter XXX. — *Of a Wheel rolling on the inside of a Hoop.* — 1. When a circle rolls on the inside of another circle, instead of on the outside, the curve is called a hypocycloid. 2. Suppose your slate-pencil were straight, so that it would lie flat on the slate, and make a mark as broad as the pencil is long. Then suppose you were to put your pencil across one corner of your slate like a hypotenuse, and slide first one end up to the corner, and then the other, keeping both ends all the time touching the slate frame. You would make a white mark in the corner, of a curved three-cornered shape, like this figure. The curve inside would be a hypocycloid. 3. If you take the corner of your slate for a centre, and the length of your pencil for a radius, and draw a quarter of a circle, as I have done in the last figure; if you then roll on the inside of this arc a circle whose diameter is one half the length of the pencil, it will make the same hypocycloid. I have also drawn this circle in the figure. 4. You can draw a hypocycloid by rolling the spool and the pencil on the inside of any little hoop held firmly on the slate. The rim of the cover of a large wooden pill-box will make a nice little hoop for this purpose. 5. The evolute of a hypocycloid is a larger hypocycloid of the same shape on the outside of it; and the hypocycloid itself may be fancied as the evolute of a smaller hypocycloid within it; so that hypocycloids, like epicycloids, are packed one within the other, like nests of tubs or boxes. 6. The hypocycloid that is made when the diameter of the rolling circle is one quarter of the diameter of the circle that it rolls in, can be made by sliding the hypotenuse backwards and forwards on the legs of a right triangle. The hypotenuse must be kept of the same length, and it will always be a tangent to the hypocycloid. 7. This hypocycloid may be called a hypocycloid of four arches; because, as you may see in the figure, both it and its evolutes have each four arches. 8. If the diameter of the spool is nearly half that of the hoop, the pencil will move across the hoop in a very flat curve, almost like a diameter of the hoop; and the evolute at the ends of the curve will be almost like two parallel straight lines at right angles to the end of the diameter; so that the string unwrapping from the evolute will be very long. When the diameter of the spool is exactly half that of the hoop, the hypocycloid is a straight line; and the evolute of it, if you can fancy that







there is any evolute, is two parallel straight lines at right angles to its ends. 9. If the diameter of the spool is more than half that of the hoop, it will make a hypocycloid like that made by a smaller spool. If you have two spools, one of them as much wider than the radius of the hoop as the other is smaller, so that the hoop will just let the two spools stand in it side by side, then one spool will make exactly the same hypocycloid as the other.



10. These two spools cannot, of course, be both rolling in the hoop at the same time; but we can easily imagine two circles of the same size as the spools rolling in a circle as large as the hoop. Start the circles from the position in which I have drawn them to rolling in opposite directions, and if you roll the little circle faster than the large one, so as to make them get round the hoop in the same time, the points in the two circles which are now touching will keep together all the time, making the same hypocycloid.

# Editorial Items.

*The following gentlemen have sent us Solutions of the Prize Problems in the December No. of the Monthly.*

ARTHUR WILKINSON, Junior Class, Harvard College, answered all the questions but I.

GEORGE B. HICKS, Student, Cleveland, Ohio, answered all the questions.

DAVID TROWBRIDGE, Student, Perry City, Schuyler Co., N. Y., answered all the questions.

JAMES M. INGALLS, Delton, Sauk Co., Wisconsin, answered questions IV. and V.

W. H. STIRLING, Student, Baltimore, Md., answered problem I.; also sent solutions of problem II. in October No., problem III. in November No., and problem I. in January No.

ASHER B. EVANS, Junior Class, Madison Univ., Hamilton, N. Y., answered all the questions.

G. W. HILL, Senior Class, Rutger's College, New Brunswick, N. J., answered all the questions.

A Student in Baltimore College answered all the questions. (RICHARD COTTER, Prof.)

E. P. AUSTIN, Student in the Univ. of Michigan, answered all but I. and III.

We shall publish the report of the Judges, with the Prize Solutions, in the next Number.

CONTENTS *of the Nouvelles Annales de Mathématiques*, January, 1859.

Fundamental Formulas of Spherical Analysis; by M. VANNSON. Upon Interpolation; by M. EUGÈNE ROUCHÉ. Application of the New Analysis (determinants) to Surfaces of the Second Order; by M. PAINVIN. Note by M. LEBESGUE. Questions Proposed; by M. VANNSON. Focal Properties of Surfaces of the Second Order, after M. HELLERMANN; by M. DEWULF.

BULLETIN *de Bibliographie, d'Histoire et de Biographie Mathématiques.*

Problem of the Young Girls, of the Blind, &c. Theorem of WARING, upon Prime Numbers. The Mathematical Monthly. Journal für die reine und angewandte Mathematik.

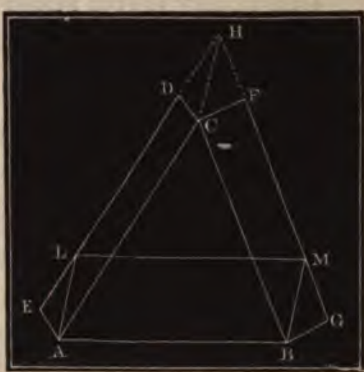
THE

# MATHEMATICAL MONTHLY.

Vol. I....APRIL, 1859....No. VII.

## PRIZE PROBLEMS FOR STUDENTS.

### I.




On two sides $A\,C$ and $B\,C$ of any triangle $A\,B\,C$, let any parallelograms $A\,C\,D\,E$ and $B\,C\,F\,G$ be described. Let $E\,D$ and $F\,G$ produced meet in $H$; join $H\,C$, and through $A$ and $B$ draw $A\,L$ and $B\,M$ equal and parallel to $H\,C$. Join $L\,M$. It is required to prove that the parallelogram $A\,L\,M\,B$ on the side $A\,B$ is equal to the sum of the parallelograms on the sides $A\,C$ and $B\,C$.

Show also that the Pythagorean proposition is a particular case of this proposition.

### II.

The area of a right-angled triangle is equivalent to the rectangle of the differences between the radius of the inscribed circle and the two shorter sides respectively; or the rectangle of the segments of the hypothenuse made by a perpendicular let fall upon it from the centre of the inscribed circle.

III.

The distance between two points $A$ and $B$ is $a$ miles. A person starts at $A$ and travels the first day one $m$th his distance to $B$; the second day he travels back one $m$th his distance to $A$; the third day he turns and travels one $m$th his distance to $B$, and so on. How far will he travel in $n$ days, and how far will he be from $A$?

IV.

The volume of any right cone equals the product of its whole surface by one third the radius of the inscribed sphere.

V.

One pulley drives another by means of a belt; given the length of the belt $l$, the diameter $D$, of the larger pulley, the distance $a$ between the centres of the pulleys; to find the diameter $d$, of the smaller pulley. Find also a simple approximate formula for the use of machinists.

## REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN NO. III. VOL. I.

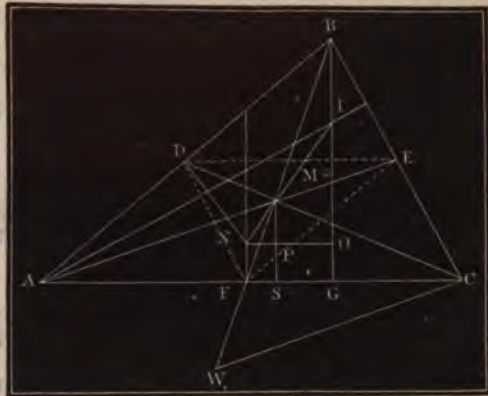
The first Prize is awarded to Asher B. Evans, of the Junior Class in Madison University, Hamilton, N. Y.

The second Prize is awarded to David Trowbridge, Perry City, Schuyler Co., N. Y.

### PRIZE SOLUTION OF PROBLEM I.

"In every triangle, the perpendiculars dropped from the angles to the opposite sides meet in the same point; the perpendiculars erected to the middle of its sides meet in the same point; the lines drawn from the angles to the middle of the opposite sides meet in the same point. Prove that these three points are in the same straight line, and that their distances apart are in a constant ratio."

Let $ABC$ be any triangle; and let $I, N, M$ be the three points. Produce $BF$ to $W$ and draw $CW$ parallel to $AE$; then will $BM = MW$, since $BE = EC$. But the triangles $AMF$ and $FCW$ are equal; hence $MF = FW$. Therefore $BM : MF :: 2 : 1$. Again the triangle $ABC$ is similar to $DEF$, and the homologous sides are as $2 : 1$. But $BI$ and $NF$ are




evidently homologous lines in these two triangles, and are also to each other as $2 : 1$. Therefore $BM : MF :: BI : NF$. But the triangles $BMI$ and $NMF$ have the angle $MBI = NFM$; and since the sides containing these equal angles are proportional, these two triangles are similar. The sides $IM$ and $MN$ are therefore in constant ratio of $2 : 1$; and since the angle $BMI = NMF$, and $BMF$ is a straight line, $IMN$ is a straight line also. The additional lines in the figure will suggest other ways of demonstrating the proposition.

This solution is by Asher B. Evans, with a slight abridgment.

PRIZE SOLUTION OF PROBLEM II.

"Prove that $\tan 9^\circ = \sqrt{5} + 1 - \sqrt{5 + 2\sqrt{5}}$."

By the simple formulas of Trigonometry we have

$$\tan 5\theta = \frac{\tan\theta + \tan 4\theta}{1 - \tan\theta \tan 4\theta};\ \tan 4\theta = \frac{2\tan 2\theta}{1 - \tan^2 2\theta};\ \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Put $\tan\theta = x$; then

$$\tan 2\theta = \frac{2x}{1 - x^2},\quad \tan 4\theta = \frac{4x(1 - x^2)}{(1 - x^2)^2 - 4x^2},$$

$$\tan 5\theta = \frac{x(1 - x^2)^2 - 4x^3 + 4x(1 - x^2)}{(1 - x^2)^2 - 4x^2 - 4x^2(1 - x^2)} = 1, \text{ when } \theta = 9^\circ.$$

By reduction

$$1 - 5x - 10x^2 + 10x^3 + 5x^4 - x^5 = 0.$$

Adding $20x^2 - 20x^3$ to both members of this equation it becomes

$$1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 = (1-x)^5 = 20x^2(1-x).$$

Hence, dividing by $1-x$ we get

$$(1-x)^4 = 20x^2;\ \text{or}\ (1-x)^2 = \pm 2x\sqrt{5};$$

and readily find that the five roots are

(1) $x = 1 = \tan 45°$,

(2) $x = 1 + \sqrt{5} + \sqrt{5 + 2\sqrt{5}} = \tan 81°$,

(3) $x = 1 + \sqrt{5} - \sqrt{5 + 2\sqrt{5}} = \tan 9°$,

(4) $x = 1 - \sqrt{5} + \sqrt{5 - 2\sqrt{5}} = \tan(-27°)$,

(5) $x = 1 - \sqrt{5} - \sqrt{5 - 2\sqrt{5}} = \tan(-63°)$.

The tan 9° is positive and less than unity, and (3) is the only root which fulfils these conditions. (2) and (3) are reciprocals, and therefore the corresponding arcs are complimentary. The same is true of (3) and (4).

This interesting solution is by David Trowbridge. The solution by E. P. Austin involves the same method, but is less complete.

PRIZE SOLUTION OF PROBLEM III.

"Find the coefficient of $x^n$ in the expansion of $\frac{ax+b}{(x-1)(x+2)(x-3)}$ into a series of ascending powers of $x$."

$$\text{Put}\ \frac{ax+b}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$
$$= \frac{(A+B+C)x^2 - (A+4B-C)x - (6A-3B+2C)}{(x-1)(x+2)(x-3)}.$$

By the theory of indeterminate coefficients

$A + B + C = 0$, $A + 4B - C = -a$, $6A - 3B + 2C = -b$,

and these equations give

$$A = -\frac{a+b}{6},\ B = \frac{b-2a}{15},\ C = \frac{3a+b}{10}.$$

Therefore by substitution we get

$$\frac{ax+b}{(x-1)(x+2)(x-3)}=\frac{a+b}{6}\times\frac{1}{1-x}+\frac{b-2a}{30}\times\frac{1}{1+\frac{x}{2}}-\frac{3a+b}{30}\times\frac{1}{1-\frac{x}{3}}.$$

Now develop each of the fractions involving $x$ by division, and we readily find

$$\frac{a+b}{6}\pm\frac{b-2a}{30\,.\,2^n}-\frac{b+3a}{30\,.\,3^n}$$

to be the required coefficient; the sign of the middle term being $+$ when $n$ is even and $-$ when it is odd.

This solution is by Asher B. Evans.

### PRIZE SOLUTION OF PROBLEM IV.

"The equations (1) $y=ax+a+1$, (2) $y=(a+1)x+b$, (3) $y=bx+5$, are equations of straight lines in which $a$ and $b$ are positive integers, and the three lines all meet in one point. What values can $a$ and $b$ have, and what are the corresponding coördinates of the common points of the lines?"

From (1) and (2) $x=a+1-b$, and from (2) and (3) $(a+1-b)\,x=5-b$; and eliminating $x$ from these equations we get $(a-b+1)^2=5-b$. Therefore

$$a=-1+b\pm\sqrt{5-b};$$

which, with the given conditions, can only be satisfied by making $b=1, 4, 5$.

The following sets of corresponding values are easily found:

$$\left.\begin{matrix}b=1\\a=2\\x=2\\y=7\end{matrix}\right\}\quad\left.\begin{matrix}a=2\\b=4\\x=-1\\y=1\end{matrix}\right\}\quad\left.\begin{matrix}a=4\\b=4\\x=1\\y=9\end{matrix}\right\}\quad\left.\begin{matrix}a=4\\b=5\\x=0\\y=5\end{matrix}\right\}.$$

For the last set of values lines (2) and (3) coincide. This solution is by E. P. Austin.

### PRIZE SOLUTION OF PROBLEM V.

"When the angle between the coördinate axes is $\frac{\pi}{3}$; prove that $x+y-a=\sqrt{xy}$ is the equation of a circle, of which the radius is $\frac{1}{3}a\sqrt{3}$."

Let this equation be transformed to a system of rectangular coördinates, having the same origin and axis of $x$. The equations of transformation from oblique to rectangular coördinates are (see J. M. Peirce's Analytic Geometry)

$$x = \frac{x_0 + x_1 \sin\left(\tfrac{y}{x} - \tfrac{x_1}{x}\right) - y_1 \cos\left(\tfrac{y}{x} - \tfrac{x_1}{x}\right)}{\sin \tfrac{y}{x}}$$

$$y = \frac{y_0 + x_1 \sin \tfrac{x_1}{x} + y_1 \cos \tfrac{x_1}{x}}{\sin \tfrac{y}{x}}.$$

In this case $\tfrac{x_1}{x} = 0, \tfrac{y}{x} = \tfrac{1}{3}\pi, x_0 = 0, y_0 = 0$, and thee quations of transformation become $x = x_1 - y_1 \sqrt{\tfrac{1}{3}}, y = 2 y_1 \sqrt{\tfrac{1}{3}}$. Substituting these values of $x$ and $y$ in the square of the given equation, it becomes

$$x_1^2 + y_1^2 - 2 a x_1 - 2 a y_1 \sqrt{\tfrac{1}{3}} + a^2 = 0; \text{ or}$$

$$(x_1 - a)^2 + (y_1 - a \sqrt{\tfrac{1}{3}})^2 = \tfrac{1}{3} a^2;$$

which is the equation of a circle, of which the coördinates of the centre are $x_0 = a, y_0 = a \sqrt{\tfrac{1}{3}}$, and radius $= \tfrac{1}{3} a \sqrt{3}$. This solution is by Arthur Wilkinson.

JOSEPH WINLOCK,
CHAUNCEY WRIGHT,
TRUMAN HENRY SAFFORD.

## PROPOSITIONS RELATING TO THE RIGHT-ANGLED TRIANGLE.

BY DAVID W. HOYT,
Adjunct Prof. of Mathematics and Mechanics in the Polytechnic College, Philadelphia.

1. Pythagorean Proposition. This can be demonstrated by proving the square described on the hypothenuse, and the sum of the squares described on the other two sides, each equal to four times the triangle, added to the square of the difference of the two shorter sides. In the annexed figure $A\,O$ and $D\,L$ are each made equal to


$BC$. The rectangle $ONLD$ and the square $BCGF$ form the rectangle $BCPI$, which is equal to $ABMO$, or $RABC$; and $LEMN$ is the square of the difference of $AB$ and $BC$. The remainder of the proof is sufficiently evident from the figure.

The above demonstration is deduced from the common method of cutting the squares $BD$ and $BG$ into five parts, which may be so combined as to form the square $AK$. This division is effected by drawing a line through $B$, parallel to $AC$, and erecting a perpendicular to $AD$ at its extremity $O$.

2. Having given the hypothenuse, $h$, and the difference $d$, of the two shorter sides, the area of a right-angled triangle is expressed by $\frac{h^2-d^2}{4}$. This is merely a corollary of the preceding demonstration.

3. Having given the hypothenuse, $h$, and the radius of the inscribed circle, $r$, the area of a right-angled triangle is expressed by $hr + r^2$.

4. The area of a right-angled triangle is equivalent to the rectangle of the differences between the radius of the inscribed circle and the two shorter sides respectively; or the rectangle of the segments of the hypothenuse made by a perpendicular let fall upon it from the centre of the inscribed circle.

5. If a straight line be drawn from the vertex of either acute angle of a right-angled triangle to the centre of the inscribed circle, the square of this line, added to the rectangle of the opposite side and the hypothenuse, will be equivalent to the square of the hypothenuse.

6. The hypothenuse of a right-angled triangle is to the sum of the two lines drawn from the vertices of the two acute angles to the centre of the inscribed circle, as the difference of those lines is to the difference of the other two sides of the triangle.

7. The square of the diameter of a circle inscribed in a right-angled triangle is equivalent to twice the rectangle of the differences between the diameter and the two shorter sides respectively.

8. The rectangle of the lines drawn from the vertices of the two acute angles of a right-angled triangle to the centre of the inscribed circle, is to the rectangle of the hypothenuse and radius, as $\sqrt{2}$ to 1.

The above propositions can be demonstrated by any one having a knowledge of the principles contained in the first four books of LEGENDRE. They are given in the hope that they may be of service to those teachers who wish to make a practical application of the remarks contained in former numbers of this Monthly concerning the study of Geometry. The last six theorems are intimately connected with each other, and with the principle that the sum of the hypothenuse and diameter of the inscribed circle is equal to the sum of the other two sides.

## PROPOSITIONS RELATING TO A PARTICULAR CONE.

BY DANIEL KIRKWOOD, LL. D.,
Prof. of Mathematics and Civil Engineering in Indiana University, Bloomington.

IN the fourth number of this Journal Professor SNELL states several curious properties of the cone whose slant height has to the radius of its base the ratio of 3 to 1. The following propositions in regard to the cone whose slant height is to the radius of its base in the ratio of $\sqrt{3}$ to 1 are equally interesting:

1. The right cone, whose slant height is to the radius of the base as $\sqrt{3}:1$, has *greater convex surface* than any other cone inscribed in the same sphere.

2. It has *greater volume* than any other cone inscribed in the same sphere.

3. Its base divides the circumscribed sphere into two segments such that the curve surface of one is *twice* that of the other.

4. Its curve surface is to that of the greater segment into which its base divides the circumscribed sphere, as the radius of the base is to the slant height.

5. Its volume is equal to that of the sphere, of which its altitude is the diameter

6. Its slant height is equal to the base of the *greatest parabola* that can be cut from it.

7. The area of such maximum parabola is to the curve surface of the lower segment of the sphere, as the diameter of a circle is to its circumference, or as $1:\pi$.

8. The *curve surface* of its greatest inscribed cylinder is to the *base* of such cylinder, as the diagonal of a square is to the side, or as $\sqrt{2}:1$.

9. Its centre of gravity is the centre of the circumscribed sphere.

10. Its altitude is to its slant height as $\sqrt{2}:\sqrt{3}$.

11. Its altitude is to the diameter of the circumscribed sphere, as $2:3$.

12. Its base is to that of the least cone that can be described about its circumscribed sphere, as $2^2:3^2$.

13. Its volume is to that of the circumscribed sphere, as $2^3:3^3$.

14. Its centre of gravity coincides with that of the *least cone* that can be described about its circumscribed sphere.

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## NOTES ON THE THEORY OF PROBABILITIES.

By Simon Newcomb, Nautical Almanac Office, Cambridge, Mass.

9. The theorem given in § 8 may be extended to any number of independent events. Let there be three events, $E_1$, $E_2$, $E_3$, of which

the respective probabilities are $p_1, p_2, p_3$. Call $p'$ the probability of the compound event $(E_1 E_2)$. Then by the theorem the probability that this compound event and the event $E_3$ will *both* occur will be $p' \times p_3 = p_1 \times p_2 \times p_3$, since $p' = p_1 \times p_2$. By repeating the same process it may be shown that in general if there are $n$ events, of which the probabilities are $p_1, p_2, p_3 \ldots . p_n$, the probability of the concurrence of all these events is the continued product $p_1 \times p_2 \times \ldots . p_n$. If the $p$'s are mostly small fractions, and $n$ large, this product will be very small.

10. If the events are not entirely independent, that is, if the probability of the second is affected by the occurrence or non-occurrence of the first; then, the probability of the compound event is equal to the probability of the first multiplied into the probability of the second *on the supposition that the first occurs.*

Let $c$ be the whole number of possible cases,—let $p$ of these cases be favorable to the first event, and of these $p$ cases let $q$ be favorable to the second. The probability that both events will occur is then $\frac{q}{c}$. But $\frac{p}{c}$ is the probability of the first event, and if this event occurs, the probability of the second becomes $\frac{q}{p}$, the product of which into $\frac{p}{c}$ gives $\frac{q}{c}$, which proves the proposition.

It may be remarked that this theorem applies also to independent events, and includes that of § 8, the probability of the second event being then the same, whether the first event be supposed to occur or fail.

Example. What is the probability that out of a bag which contains two white and two black balls, a white ball will be drawn at the first drawing and a black ball at the second? In this case, if the first event occur, the probability of the second will be $\frac{2}{3}$, since two out of the three balls remaining will be black. The probability of the first event being $\frac{1}{2}$, the probability of the compound event will be $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$.

11. *Conflicting events.* When a system of events are so connected that one *must* and only one *can* occur, they are called a *system of conflicting events.*

The sum of the probabilities of a system of conflicting events is equal to unity; since, every separate case being favorable to one of them, and to one only, the sum of the numerators of the different fractions expressing the probabilities (§ 6) is equal to the common denominator.

12. *Value of expectation.* Let there be $n$ men, $q$ of whom to be determined by lot, are certain of receiving the sum $s$ each. What is each man's chance worth? If each man were to sell out his chance, the purchaser should give in all the sum $q\,s$ which he would then be sure of receiving, and since each man's chance is equal, he ought to give each man $\frac{q}{n}s$. But $\frac{q}{n}$ is the probability for each man that he will receive the sum $s$, and the value of his chance is therefore equal to the sum that he may receive multiplied by the probability that he will receive it.

## PROBLEMS IN PROBABILITIES.

*A* HAS the reputation of telling the truth as often as three times in four; *B* as often as four times in five; and *C* as often as five times in seven. When *A* and *B* agree in affirming what *C* denies, what is the probability that *A* and *B* tell the truth? — Communicated by Professor ROOT.

A person goes on throwing a common die until he throws an ace; at whatever throw this occurs (the $n$th) he is to receive the $n$th of a dollar. What is the value of his expectation? — Communicated by ASHER B. EVANS.

## DEMONSTRATION OF THE PROPOSITIONS RELATING TO THE CONE AND SPHERE.*

BY E. S. SNELL,
Prof. of Mathematics in Amherst College, Amherst, Mass.

1. To find the greatest cone within a given superficies. Let $r =$ radius of base, and $s =$ slant height. Then, perimeter of base $= 2\pi r$; area of base $= \pi r^2$; convex surface $= \pi r s$. Therefore the given area

$$A = \pi r^2 + \pi r s. \quad \therefore s = \frac{A}{\pi r} - r:$$

and the volume

$$v = \frac{\pi r^2}{3}(s^2 - r^2)^{\frac{1}{2}} = \frac{\pi r^2}{3}\left(\frac{A^2}{\pi^2 r^2} - \frac{2A}{\pi}\right)^{\frac{1}{2}},$$

which is to be a maximum. For convenient differentiation, square this function, reject the constant factors, and we have $A^2 r^2 - 2\pi A r^4$. Its differential coefficient

$$2A^2 r - 8\pi A r^3 = 0. \quad \therefore r = \tfrac{1}{2}\sqrt{\tfrac{A}{\pi}}.$$

$$\text{But, } s = \frac{A}{\pi r} - r = \frac{A}{\pi} \div \tfrac{1}{2}\sqrt{\frac{A}{\pi}} - \tfrac{1}{2}\sqrt{\frac{A}{\pi}} = \tfrac{3}{2}\sqrt{\frac{A}{\pi}} = 3r.$$


Therefore the slant height equals three times the radius of the base. This proves the truth of proposition 1.

2. To find the cone of least *entire surface*, circumscribing a given sphere.

Let the figure represent the section of the cone and sphere through the axis. Let $FC = a$, and $AC = x$. Hence, $AB = a + x$, and $AF = (x^2 - a^2)^{\frac{1}{2}}$. $AFC$ and $ABG$ are similar triangles;

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* In No. IV. p. 121, 122.

$\therefore (x^2 - a^2)^{\frac{1}{2}} : x :: x + a : A\,G = \frac{x\,(x + a)}{(x^2 - a^2)^{\frac{1}{2}}} =$ slant height.

$\therefore (x^2 - a^2)^{\frac{1}{2}} : a :: x + a : B\,G = \frac{a\,(x + a)}{(x^2 - a^2)^{\frac{1}{2}}} =$ radius of base.

$\therefore \frac{2\,\pi\,a\,(x + a)}{(x^2 - a^2)^{\frac{1}{2}}} =$ perimeter of base.

$\therefore \frac{\pi\,a^2\,(x + a)^2}{x^2 - a^2} = \frac{\pi\,a^2\,(x + a)}{x - a} =$ area of base.

$\therefore \frac{x\,(x + a)}{(x^2 - a^2)^{\frac{1}{2}}} \times \frac{\pi\,a\,(x + a)}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{\pi\,a\,x\,(x + a)}{x - a} =$ convex surface.

$\therefore \frac{\pi\,a^2\,(x + a)}{x - a} + \frac{\pi\,a\,x\,(x + a)}{x - a} = \frac{\pi\,a\,(x + a)^2}{x - a} =$ whole surface, which is to be a minimum. The differential coefficient is

$$\pi\,a\left(\frac{2\,(x + a)\,(x - a) - (x + a)^2}{(x - a)^2}\right) = 0;\ \text{and}\ x = 3\,a.$$

Therefore $A\,B = 4\,a =$ twice the height of the sphere. Substituting $3\,a$ for $x$ in the expression for surface, $\frac{\pi\,a\,(x + a)^2}{x - a}$, we have $8\,\pi\,a^2$. But the surface of the sphere is $4\,\pi\,a^2$; therefore, the entire surface of the required cone is *twice* that of the sphere. Slant height : rad. of base $:: x : a :: 3\,a : a :: 3 : 1$. This cone, therefore, has the same form as that in the first demonstration; and thus, propositions 2, 4, and 5 are proved.

3. To find the cone of least *volume*, circumscribing a given sphere.

As before, area of base $= \frac{\pi\,a^2\,(x + a)}{x - a}$, and height of cone $= x + a$. Therefore the volume $= \frac{\pi\,a^2\,(x + a)^2}{3\,(x - a)}$, which is to be a minimum. This function differs from that for the entire surface, only in having the constant factor $\frac{1}{3}\,a^2$ in place of $a$. Hence, as before, $x = 3\,a$. This cone is, therefore, identical with that in the second demonstration. The volume, $\frac{\pi\,a^2\,(x + a)^2}{3\,(x - a)} = \frac{16\,\pi\,a^4}{6\,a} = \frac{1}{3}\,\pi\,(2\,a)^3$. But the volume of the sphere $= \frac{1}{6}\,\pi\,(2\,a)^3$; therefore, the volume of the

cone is *twice* that of the inscribed sphere. Thus have been proved propositions 3 and 6.

4. To find in what ratio the spheric surface is divided by the circle of contact.

$A\,C$ has been proved equal to $3\,C\,F$. But, $A\,C : C\,F :: C\,F : E\,C :: 3 : 1$; therefore $B\,E : E\,D :: 3 + 1 : 3 - 1 :: 2 : 1$; and the surfaces, $F\,B\,H : F\,D\,H :: 2 : 1$; which is proposition 7.

5. Proposition 8 was accidentally stated wrong; it should have been, "The surface of the inscribed sphere is *twice* the area of the base of the cone." To prove it, we have only to compare the expressions for the two in the second demonstration.

The area of the base $= \frac{\pi a^2 (x + a)}{x - a} = 2 \pi a^2$. But the surface of the sphere $= 4 \pi a^2$, or twice the base of the cone.

*Remark.* Instead of eliminating $s$ from the volume, as Professor SNELL has done, we may prove proposition 1 as follows:

Since the whole surface $= \pi r^2 + \pi r s =$ constant, and the volume $= \frac{\pi r^2}{3} (s^2 - r^2)^{\frac{1}{2}} =$ maximum, the derivatives of both functions must equal zero. Removing constant factors and radicals for convenience, we have

$$r^2 + r s = \text{constant},$$
$$r^4 s^2 - r^6 = \text{maximum}.$$

The derivatives are

$$2 r + s + r D_r s = 0,$$
$$4 r^3 s^2 + 2 r^4 s D_r s - 6 r^5 = 0.$$

Eliminating $D_r s$ from these equations, we obtain

$$s^2 - 2 s r - 3 r^2 = 0;$$

and, therefore, $s = 3 r$, or $- r$; one root corresponding to a maximum and the other to a minimum.

JAMES CLARK, Esq., of Wayne, Maine, sent us demonstrations of the above propositions, calling attention to the oversight in proposition 8, which Professor SNELL has corrected.

## TO DESCRIBE A CIRCLE TANGENT TO THREE GIVEN CIRCLES.

BY H. A. NEWTON,
Professor of Mathematics, Yale College, New Haven, Conn.

The method of transformation of curves by reciprocal radii vectores* affords a ready solution of the problem of drawing circles tangent to three given circles, provided two of the given circles cut each other.

This method is as follows. Let any point $P$ be taken and called the *pole*, and any arbitrary square be taken as a superficial unit.

Two points, $A$ and $B$, are said to be *reciprocal* to each other if they are on the same straight line through $P$ and the rectangle $PA.PB$ is equal to the assumed unit square.

Two curves are said to be *reciprocal* to each other when the reciprocal to each point of the one is on the other.

*The reciprocal of a straight line is a circumference that passes through the pole.*

Fig. 1.

Let $AB$ (Fig. 1) be a straight line, and $P$ the pole. Let fall $PA$ perpendicular on $AB$, and let $C$ be the reciprocal to $A$. On $CP$ as a diameter describe a circle $CFP$. From $D$ any point of $AB$ draw $DP$ and take $E$ the reciprocal of $D$. $E$ is in the circumference $CFP$. For, join $CE$; then since by hypothesis $PA.PC = PD.PE$, that is $PA:PD::PE:PC$, the triangles $PEC$ and $PAD$ are similar, and the angle $PEC = PAD$ a right

* Lionville XII. 265, and XIII. 209. Salmon's Higher Plane Curves, p. 239. Serret Méthodes en Géométrie, p. 21. Camb. and Dub. Math. Journal, Feb. 1853. Quarterly Jour. of Pure and Applied Math. 1. 32.

angle. Hence $E$ is in the circumference $PFC$, and the circumference $CFP$ is the reciprocal of the straight line $AB$.

Cor. 1. A straight line through $P$ is reciprocal to itself.

Cor. 2. Circumferences reciprocal to parallel straight lines are tangent to each other at the pole.

*The reciprocal to the circumference of a circle is a circumference.*


Fig. 2.

Let $ABC$ (Fig. 2) be a circumference, and $BA$ the diameter through $P$. Take $a$ and $b$ points reciprocal to $A$ and $B$, and on $ab$ as a diameter describe the circle $adb$. These two circumferences are reciprocal to each other. For let $C$ be any point of $ABC$, and $c$ its reciprocal point. Join $CP$, $CA$, $CB$, $ca$, and $cb$. Since $PB \, . \, Pb = PC \, . \, Pc$ the triangles $PBC$ and $Pbc$ are similar, and the angle $PBC = Pcb$. For a like reason $PAC = Pca$. Therefore $Pcb + Pca$ or $acb$ is equal to $BAC + ABC$ a right angle, and $c$ is in the circumference $adb$.

If $P$ is without the circle $ABC$, the demonstration is nearly the same.

*If two curves cut each other at any angle, their reciprocals cut each other at an equal angle.*


Fig. 3.

Let the two curves (Fig. 3) intersect in $A$. Their reciprocals will meet in $B$ the reciprocal of $A$. If $n$ be a point in one curve and $m$ its reciprocal on the reciprocal curve, a circumference may be described through $A$, $B$, $m$, and $n$, for $PA \, . \, PB = Pm \, . \, Pn$. If $m$ and $n$ approach $B$ and $A$ and coincide with them, the circle $ABmn$ becomes tangent to one curve at $A$ and to

its reciprocal at $B$. A second circle may be tangent to the second curve at $A$ and to its reciprocal at $B$. Now the angle made by these circumferences at $A$ is evidently equal to that at $B$. Therefore the curves cut each other at the same angle at $A$ as their reciprocals at $B$.

If now a figure consists of curves and points, and a second figure consists of the reciprocals of these curves and points, and if in the first figure there be proved any proposition or construction, we may, in many cases, infer some corresponding proposition or construction in the second figure.

A few examples will best illustrate this.

All the straight lines that cut a given circumference at right angles pass through the centre.

All circumferences that pass through a given point $P$ and cut a given circumference at right angles, pass through a second common point, which may be called the *reciprocal centre* with respect to $P$.

All the straight lines that cut a given circumference at the same oblique angle, are tangent to a second circle concentric with the given circle.

All the circumferences that pass through a fixed point $P$, and cut a given circumference at the same oblique angle, are tangent to a circle.

Two straight lines may be drawn parallel to a given straight line, and tangent to a given circle.

Two circles may be drawn to touch a given circle at a given point $P$, and also tangent to a second given circle.

To construct the points of tangency, draw through the centre of the given circle a straight line at right angles to the given straight line. It will meet the circumference in the required points.

To construct the points of tangency, draw through the given point $P$ and the reciprocal centre of the second circle with respect to $P$, a circumference cutting the first circumference at right angles. It will meet the second circumference in the required points.

If through three points $A$, $B$, and $C$, taken two and two, three straight lines, $A B$, $B C$, and $A C$, and also the circle $A B C$ be drawn, then will any one of these straight lines cut the circumference $A B C$ at an

If through a point $P$ and through three points $A$, $B$, and $C$, taken two and two, three circumferences $P A B$, $P B C$, and $P A C$ and also the circumference $A B C$ be drawn, then will any one of the circum-

angle equal to that made by the other two straight lines. — (Euc. 32. 3.)

ferences that pass through $P$ cut $ABC$ at an angle equal to that made by the other two circumferences.

This is the fourth prize problem in the Jan. No. of the Monthly.

If three circles be given, two of which intersect, and one of the points of intersection be taken as pole, the reciprocal figure will consist of two straight lines and a circle. To draw a circle tangent to three given circles, corresponds to drawing a circle tangent to two given straight lines and a given circle.

We may do this by the aid of the following Lemma.

LEMMA.

If in any plane there be given two straight lines and a circle, and if about the circle there be described a parallelogram whose sides are parallel to the two given straight lines, and if four straight lines be drawn from the four angles of this parallelogram to the intersection of the given straight lines, then through each point where any one of these four lines cuts the circumference of the given circle, can a circle be described to touch both the given straight lines and the given circle.

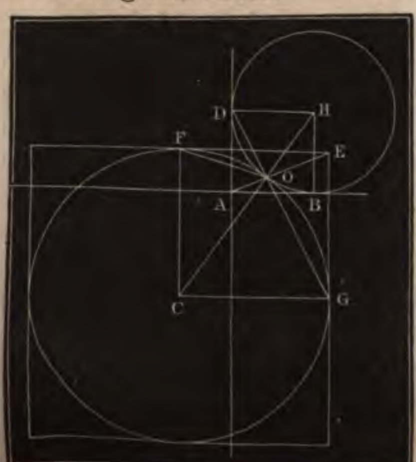


Fig. 4.

Let $C$ (Fig. 4) be the centre of the given circle, $AB$ and $AD$ the given straight lines, $EF$ and $EG$ sides of the parallelogram parallel respectively to $AB$ and $AD$, and touching the circle in $F$ and $G$. Draw $AE$ cutting the circumference of the circle in $O$, $GO$ cutting $AD$ in $D$, $FO$ cutting $AB$ in $B$, $BH$ perpendicular to $AB$ and meeting a line through $C$ and $O$ in $H$. Join $CF$, $CG$, and $DH$. Since $EG$ is

parallel to $AD$, $EF$ to $AB$, and $CF$ to $BH$, we have the proportions $GO:OD::EO:OA::FO:OB::CO:OH$. Therefore $CG$ and $DH$ are parallel, and $DH$ is perpendicular to $AD$. Likewise we have $CF:CO:CG::HB:HO:HD$. But the first three terms being equal the last three are equal to each other, and a circle described from $H$ as centre with a radius $HB$ will pass through $B$, $D$, and $O$, and will evidently touch both the lines and the circle in these points.

It may be easily shown, that through no point of the circumference of the given circle other than those on the four lines drawn as above can a circle be described to touch both the given lines and the given circle.

We may now construct the points of tangency of all the circles that can touch two given straight lines and a given circle, and hence also the points of tangency of all the circles which touch three given circles, two of which must, however, intersect.

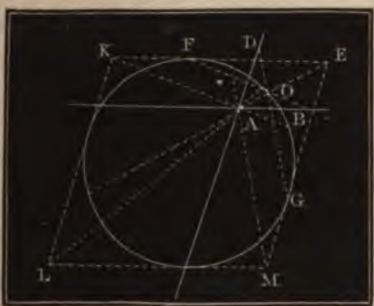



Fig. 5.

Let $AB$ and $AD$ (Fig. 5) be the two straight lines, and $GOF$ the given circle. Draw two straight lines parallel to the straight line $AB$ touching the circle $GOF$; also two parallel to $AD$ and tangent to $GOF$. Through $A$ and each one of the four points of intersection, $E$, $K$, $L$,


Fig. 6.

Let $PAB$ and $PAD$ (Fig. 6) be the two circles which intersect and $GOF$ the third given circle. Draw two circles touching the circle $PAB$ in $P$ and touching also the circle $GOF$; also two circles touching $PAD$ in $P$ and tangent to $GOF$. Through $P$ and $A$, and each one of the

and $M$, of these tangents draw straight lines. The points where they cut the circumference $G\,O\,F$ are the points where the required circles touch it. The points of tangency on $A\,B$ and $A\,D$ may be found thus. Let the straight line $A\,E$ determine a point of tangency $O$ on $G\,O\,F$, and let the line through $E$ parallel to $A\,D$ and tangent to $G\,O\,F$ touch $G\,O\,F$ in $G$. Draw a straight line through $G$ and $O$, cutting the line $A\,D$ in $D$. In like manner determine $B$ on $A\,B$. The circle described through $D$, $B$, and $O$ will touch in those points the given straight lines and the given circle.

four points of intersection, $E$, $K$, $L$, and $M$ of these circles, describe circles. The points where they cut the circumference $G\,O\,F$ are the points where the required circles touch it. The points of tangency on the circles $P\,A\,B$ and $P\,A\,D$ may be found thus. Let the circle $P\,A\,E$ determine a point of tangency $O$ on $G\,O\,F$, and let the circle through $E$ tangent to $P\,A\,D$ at $P$ and tangent to $G\,O\,F$ touch $G\,O\,F$ in $G$. Describe a circle through $P$, $G$, and $O$, cutting the circle $P\,A\,D$ in $D$. In like manner determine $B$ on $P\,A\,B$. The circle described through $D$, $B$, and $O$ will touch the three given circles. If one or more of the three given circles were replaced by straight lines this construction would with slight changes be applicable.

## THE MOST THOROUGH UNIFORM DISTRIBUTION OF POINTS ABOUT AN AXIS.

By Chauncey Wright, Nautical Almanac Office, Cambridge, Mass.

Let it be required to place at equal successive intervals round and round upon the circumference of a circle, an indefinite number of points, so that the circumference shall at any time be divided by them into the smallest parts; that is, so that the circumference shall be most thoroughly divided at every step of the distribution.

If we were not limited to equal intervals between the successive points, the distribution might be very simply effected either by continually bisecting or trisecting the parts of the circumference. Thus two points placed oppositely would divide the circumference into two equal parts, and with two other points these semi-circumferences might be bisected, according to the arrangement of cruciform flowers

and whorls; and further, four more points might bisect the quadrants, and so on. Or, again, the circumference might at first be divided by three points into three equal parts, and these might be bisected or trisected by the three or six following points, and so on.

To divide the parts of the circumference into smaller fractions than thirds would be to neglect the distribution at first, though the ultimate division of the circumference would be quite as perfect.

But if now we seek a uniform and symmetrical distribution as well as a thorough one, the interval between the successive points must be constant, and if the circumference is to be indefinitely subdivided, this interval is of course incommensurate.

Let $x$ denote the ratio of this interval to the whole circumference; then $\frac{1}{x}$ is the number of times the interval is contained in the whole circumference. Let $q$ denote the integer part of $\frac{1}{x}$, and $r'$ the remainder after subtracting $q\,x$ from the whole circumference.

In the second revolution around the circumference each of the parts $x$ is divided into the parts $r'$ and $x - r'$. In the third revolution each of the parts $x - r'$ into the parts $r'$ and $x - 2\,r'$, and so on as many times as $r$ is contained in $x$. Let this number be $q'$, and let the remainder, $x - q'\,r'$, be denoted by $r''$.

By further revolutions each of the intervals $r'$ is subdivided into $q''$ parts $r''$ with a third remainder $r'''$; and so on. The numbers and spaces $q$, $q'$, $q''$, &c., $r'$, $r''$, $r'''$, &c., are such as are obtained by the method for finding the greatest common divisor or for forming a continued fraction, and since

$$1 = q\,x + r', \qquad \frac{1}{x} = q + \frac{r'}{x};$$

$$x = q'\,r' + r'', \qquad \frac{x}{r'} = q' + \frac{r''}{r'};$$

$$r' = q''\,r'' + r''', \qquad \frac{r'}{r''} = q'' + \frac{r'''}{r''};$$

$$r^{[n-1]} = q^{[n]}\,r^{[n]} + r^{[n+1]}, \qquad \frac{r^{[n-1]}}{r^{[n]}} = q^{[n]} + \frac{r^{[n+1]}}{r^{[n]}};$$

we find for $x$ the value $x = \cfrac{1}{q + \cfrac{1}{q' + \cfrac{1}{q'' + \cfrac{1}{q''' + \text{\&c.}}}}}$

Now in order that the circumference may be at every step most thoroughly divided, the magnitudes of the parts into which it is divided should be as nearly equal as possible; that is, the smaller should be contained in the larger the least number of times; hence, in general, the numbers $q^{[n]}$, which express the ratios of successive sub-intervals, should be unity. It may not be necessary that the distribution should begin till after the first revolution. In this case the first quotient $q$ may be any number, but all the other quotients, $q'$, $q''$, $q'''$, &c., must be unity, hence

(1) $x = \cfrac{1}{q + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \text{\&c.}}}}}$, or putting $k = \cfrac{1}{1 + \cfrac{1}{1 + \text{\&c.}}}$, we have $x = \frac{1}{q+k}$ in which $k = \frac{1}{1+k}$; whence $k^2 = 1 - k$, or in the form of a proportion $1 : k = k : 1 - k$; that is, $k$ is the ratio of the extreme and mean proportion. Its value is $\frac{1}{2}(\sqrt{5} - 1)$. From the property which it here exhibits we may also call it the *distributive ratio.*

This ratio $k$ and the continued fraction, which expresses it, have hitherto been obtained by mathematical induction from the fractions of the Phyllotaxis. On the other hand we shall be able by further deductions, still subject to the conditions of our problem, to obtain these fractions as special solutions. For, in cases where the circumference is to be divided into a limited number of parts, the interval

$x$ becomes commensurate, and the last remainder or space of the foregoing analysis is contained exactly twice in the last but one; that is, the last step of the distribution consists in bisecting the previous parts of the circumference. Or, what comes to the same, the last remainder but one is contained in the preceding remainder once, with a remainder equal to itself, so that the last two remainders are equal. Hence, for a distributive commensurate interval, all the values $q^{[n]}$ are unity, but are limited in number; and the interval is therefore one of the approximations of the continued fraction (1).

When $q = 1$ these approximations are $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$, &c. When $q = 2$ they are $\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}$, &c., the arithmetical complements of the former, and they therefore express the same arrangements, but in an opposite direction around the circumference. When $q = 3$ we obtain $\frac{1}{3}, \frac{1}{4}, \frac{2}{7}$, &c.

In general the third approximation $\frac{1}{q+\frac{1}{1+1}} = \frac{2}{2q+1}$, or two divided by any odd number, expresses the intervals which, in the second revolution, are bisected. The fractions $\frac{2}{3}$, $\frac{2}{5}$, and $\frac{2}{7}$ are the only ones of this simplest form of distribution which are found in the arrangements of leaves around their stems. In all the arrangements of the Phyllotaxis every point after the first revolution is so placed with reference to the two points, between which it falls, that its distance from the middle is never more than one sixth of the whole interval between the two points. The distributive property of these fractions clearly explains their office in nature.

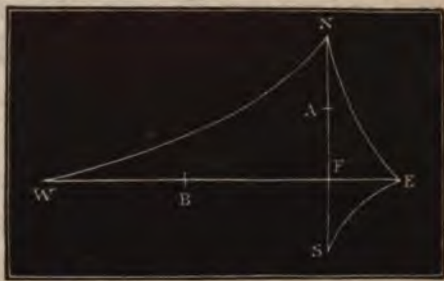
Many other fractions are apparently as well adapted to the symmetry of vegetable forms, and the limitation of natural arrangements to the Phyllotactic system seems, therefore, at first sight, unaccountable. Two hypotheses have been advanced to explain this limitation; the one attempting to deduce these arrangements mechanically from a hypothetical law of formation; and the other

regarding them as typical forms or models which nature has chosen to follow by an arbitrary limitation of her means. The latter hypothesis certainly agrees best with the principles of animal and vegetable Morphology, according to which organized forms are not determined simply by their functions, but are rather certain typical structures or models, modified and metamorphosed by their functions.

There are three ways in which we may seek to account for natural forms and phenomena. First, by deducing them from the law of their development, as in the mechanical sciences; or, secondly, we may account for them by the discovery of their functions or offices as in Physiology; or, thirdly, by referring them to fundamental types or natural methods, as in the Morphology of organized forms. To the latter class of explanations belongs the most generally received account of the Phyllotaxis; that is, the arrangements of this system are regarded as the models by which nature works, and not as the result of any discoverable law of formation, nor as performing any special function in the vegetable economy. But we have shown that there is a purpose which these arrangements fulfil, and that no other arrangements are adapted to the same office. This office in plants is to effect the most thorough distribution of leaves and petals around their axes; to expose them most effectively to the influences of light and air, and to distribute their fibres most thoroughly in the stem. Thus the functions of leaves and petals determine not only their general forms, but also in general their relative positions. Expansion and exposure are the conditions required by the functions, and the Phyllotaxis is only a simple mathematical deduction from these conditions.

## PROBLEMS IN "CURVES OF PURSUIT."

1. Three foxes are supposed to be at $F$, and a dog at $S$, forty rods south of $F$. They all start at the same instant, the foxes running with the same uniform velocity and the dog just twice as fast. The first fox runs east along the line $FE$; the second one runs north along the line $FN$; and the




third runs west along the line $FW$. The dog runs directly towards the first fox, and overtakes it at $E$. He immediately pursues the second one and overtakes it at $N$; then pursues the third and overtakes it at $W$. How far must the dog run to catch the three foxes? — Communicated by Professor Root.

The general problem, when the body pursued moves on a straight line, may be stated thus: Find the curve, such, that the intercepts of its tangents on the axis of $x$ shall always bear a constant ratio to the lengths of the arcs between the corresponding points of tangency.

2. The problem is still more general when the body pursued moves on a given curve instead of a straight line. It may be thus stated: Find the curve, such, that the intercepts of its tangents on a given curve shall always bear a constant ratio to the lengths of its arcs between the corresponding points of tangency.

When the given curve is a circle, and the required curve starts at the centre, and the ratio is unity, we have a special case which was communicated by Joseph Ficklin, Jr., Esq.

If $A$ pursues $B$, and their velocities are in the ratio of $m$ to $n$, there are obviously three cases to be considered. First, when $m > n$, $A$ will overtake $B$, and in the complete discussion $A$ must pass and

afterwards move directly away from $B$. Second, when $m = n$, $A$ will never overtake $B$, and the paths will ultimately become asymptotes to each other, except in those cases where $A$ must continue to cross and recross $B$'s path. Third, when $m < n$, $A$ will never overtake $B$; but its nearest approach is a point to be determined.

The class of curves known as "curves of pursuit" were named and first discussed by BOUGUER, in a Memoir, entitled "*Upon new Curves to which we may give the name of Lines of Pursuit*," and printed among the *Mémoires de l'Académie Royale des Sciences*, for the year 1732.

M. BOUGUER remarks, that besides the general interest which should attach to these curves as a class possessing a characteristic property, they are frequently traced, and especially at sea, where the erroneous practice prevails of always directing the prow of the vessel towards the one to which chase is given; thus sailing on the curve of pursuit instead of the straight lines of shortest distance, which will always be directed ahead of the pursued vessel. He only discusses the case in which the pursued body moves on a straight line, not referring at all to the more general problem. The Memoir immediately following that of M. BOUGUER, in the same volume, is by the illustrious M. DE MAUPERTUIS, who solves the problem as announced by BOUGUER, then states the more general one, and gives a general method for finding its differential equation; which, however, he does not integrate for any special cases.

Nor am I aware that the equation has ever been integrated even in the simplest case, that in which the given curve is a circle. When, however, the velocities are equal, as in Mr. FICKLIN's problem, the curve of pursuit continually approaches the circle, to which it becomes an asymptote, and meets only after an infinite number of revolutions, when the two bodies will be together. When $m > n$, $A$ will overtake $B$, and if $A$'s motion continues, its path outside of the

circumference will be a wave-like curve, the oscillations growing smaller as $A$'s distance from the circumference becomes greater.

In Vol. II. of the *Correspondance sur l'école polytechnique*, p. 275, we find the following statement:

"M. Dubois-Aymé se promenait sur le bord de la mer; il aperçut, à quelque distance, quelqu'un de sa connaissance, et se mit à courir pour l'atteindre; son chien, qui s'était écarté, courut vers lui, en décrivant une courbe dont l'empreinte resta sur le sable. M. Dubois, revenant sur ses pas, fut frappé de la régularité de cette courbe, et il en chercha l'équation, en supposant, 1.° que le chien se dirigeait constamment vers l'endroit où il voyait son maître; 2.° que le maître parcourait une ligne droite; 3.° que les vitesses du maître et du chien étaient uniformes."

In connection with the above, an erroneous equation of the curve is given. The next reference to the problem may be found in Gergonne's *Annales de Mathématiques*, Vol. 13, p. 145, in a Memoir entitled *Solution Nouvelle d'un problème énoncé dans la correspondance sur l'école polytechnique;* par M. Thomas de St-Laurent.

As this title indicates, there is no reference in this Memoir to the discussion of the same problem made nearly one hundred years before; and as it is called *Problème du chien,* it is to be supposed that the author did not even know the problem by its earlier name. In the same volume of Gergonne's *Annales,* p. 289, M. Thomas de St-Laurent and M. Ch. Sturm have solved the following problem, which is an extension of the one quoted above:

"Problème. Un chien, qui se trouve en un point donnè de l'un des bords d'un canal rectiligne d'une largeur constante, apercevant, en un point donné de l'autre bord, son maître qui marche le long de ce bord, avec une vitesse constante, se jette à la nage pour le joindre. En nageant, il se dirige constamment vers son maître, avec un effort toujours constant; mais le courant de l'eau, en l'entraînant, le détourne sans cesse, et avec un effort également constant, de la direction qu'il veut prendre; on demande, d'après ces diverses circonstances, quelle courbe ce chien décrira sur la surface de l'eau?"

In the Table of Contents these problems are called Problems in "Curves of Pursuit."

# A SECOND BOOK IN GEOMETRY.

## REASONING UPON FACTS.

BY THOMAS HILL.

## PREFACE.

THIS book is intended as a sequel to the "First Lessons in Geometry," and, therefore, presupposes some acquaintance with that little treatise. I think it better, however, that some interval should elapse between the study of that book and of this, — during which time the child may be occupied in the study of Arithmetic.

Geometrical facts and conceptions are easier to a child than those of Arithmetic, but arithmetical reasoning is easier than geometrical. The true scientific order in a mathematical education would therefore be, to begin with the facts of Geometry, then take both the facts and reasoning of Arithmetic and afterwards return to Geometry, not to its facts only, but to its proofs.

The object of "First Lessons in Geometry" is to develop the child's powers of conception; the object of this book is to develop his powers of reasoning. That book I consider adapted to children from six to twelve years of age, this to children from twelve to eighteen years old.

## CHAPTER I.

### PRELIMINARY.

1. GEOMETRY is the science of form. We really begin to learn Geometry when we first begin to notice the forms of things about us. Some persons observe forms much more closely than others do; partly owing to their natural taste, and partly to their peculiar education. The study of plants, animals, and minerals, the practice of drawing, and the use of building blocks and geometrical puzzles, are good modes of leading one to notice quickly and accurately differences of form.

2. The second step in learning Geometry is to become able to imagine perfect forms, without seeing them drawn. The little book called "First Lessons in Geometry" was chiefly designed to help in the attainment of this power. It is filled with descriptions of forms that cannot be exactly drawn. This is especially true of many of the curves, which cannot be drawn so exactly as straight-lined figures and circles; but which we can, with equal ease, imagine perfect.

3. The third step in learning Geometry is to learn to reason about forms, and to prove the truth of the interesting facts that we have observed. This is the only way in which we can become able to find out new truths and to be certain that they are true. And the first part of this second book is written to teach the scholar how to reason out or prove geometrical truths.

4. After learning to reason out or prove geometrical truths, it is pleasant to know how to use them. This is not the only object of Geometry. It is worth while to know a truth, simply because it is true. But it is also pleasant to be able to apply that truth to practical use, for the benefit of our fellow men. And the second part of this book is written to show in what way we can turn Geometry to practical use.

## CHAPTER II.

### DEFINITIONS.

5. **Geometry** is the science of form. Every form or shape is, in general, enclosed by a surface; every surface can be imagined as bounded, or else as divided by lines; and in every line we can imagine an endless number of points.

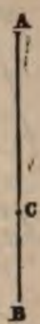
6. A point is a place without any size. It has a position, but no dimensions; neither length, breadth, nor depth.

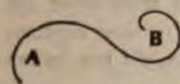
7. A line is a place having length, without breadth or depth. As we attempt to mark the position of a point by making a dot with the point of a pen or pencil, and the position of a line by moving the pencil point along the surface of the paper, we find it convenient to speak of a geometrical line as if it were made by the motion of a geometrical point. As the eye runs along the pencil line, so the eye of the mind runs along the geometrical line from point to point.

8. A surface is a place having length and breadth, without depth.

9. A solid is a place having length, breadth, and depth. A geometrical solid is not a solid body, but is simply the space that a solid body would occupy, if it were of that shape and in that place. In like manner a geometrical surface is not the surface of a solid body, but simply the surface of a geometrical solid.

10. A straight line is a line that does not bend in any part. A point moving in it never changes the direction of its motion, unless it reverses its direction.

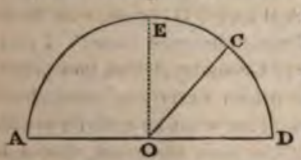



11. A curve is a line that bends very slightly at every point. It must not have any straight portion, nor any corners; that is to say, it must bend at every point, but the bend must be too small, at each place, to be perceptible.

12. A plane is a geometrical surface, such that a point, moving in a straight line from any one point in the surface to any other point, never leaves the surface. The common name of a plane is "a *flat* surface."

13. An angle is the difference of two directions in one plane. If the line C O should turn around the point O so as to make the arc D C grow larger, the difference of the directions of O C and O D would increase, and we should say that the angle D O C grew larger and larger until the point C arrived at A, so that the two lines O D and O C were opposite in direction.




14. If the point C were carried round half way to the opposite point A, that is, to the point E, the angle D O C would be a right angle, as D O E is. A right angle is a difference of direction half as large as that of oppositeness of direction. The difference between an angle and a right angle is called the complement of the angle. The difference between an angle and two right angles is called the supplement of the angle. Thus C O E is the complement of D O C, and C O A is the supplement of D O C.

15. When two lines make no angle with each other or make two right angles, they are called parallel lines. That is to say, parallel lines are those that lie in the same direction or in opposite directions. When two lines in a plane are not parallel, the point where they cross, or would cross if prolonged, is called the vertex of the angle.

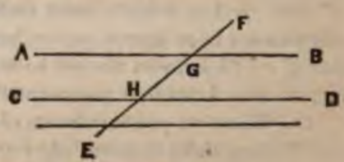
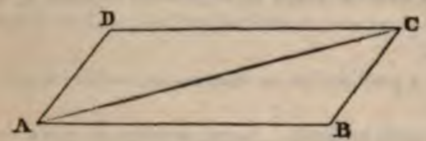


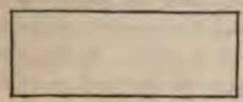

Fig. A.

16. A triangle is a figure inclosed by three straight lines in one plane.


17. A right triangle is a triangle in which two of the sides make a right angle with each other. These sides are then called the legs of the triangle, while the third side is called the hypothenuse.




18. A parallelogram is a figure bounded by four straight lines in a plane, with its opposite sides parallel.



19. A rectangle is a parallelogram with its angles all right angles.

20. A square is a rectangle with its sides all equal.

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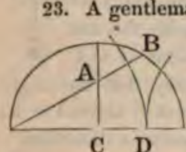
## CHAPTER III.

### REASONING.

21. SUPPOSE that we wished to make another person believe that the three angles of a triangle are, together, equal to two right angles. One way of convincing him would be to take a triangular piece of card, or of paper, cut off the corners by a waving line, and lay the three corners together, to show him that the outer edges will make a straight line, as two square corners put together will do.

22. Yet he might not be satisfied that the line was perfectly straight. Or perhaps he might say that if the angles of the triangle were in a different proportion, the corners put together would not make a straight line with their outer edges.

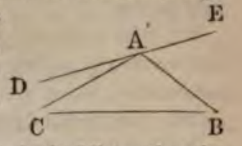
23. A gentleman once came to me and said, "I have found out that if you draw such and such lines, you will always find these two, A B and C D, equal. At least my most careful measurement shows no difference between them." I said to another gentleman, who knew something of Geometry, "Can you prove that these lines will be equal if the figure is drawn exactly as directed?" He said he would try, and in a few days he sent me what he called a proof. But on reading it I found it only amounted to saying that "if the lines are equal, they are equal." I then examined the matter myself and found that the lines were, in reality, never equal, although the difference was always very small, — too small to be easily discovered by measurement.




24. Such errors, too small to be discovered by measurement, are sometimes large enough to do great mischief; and at any rate, however small, they are still errors, and it is best to get rid of errors, and to find the exact truth, whether the error is mischievous or not. In order to do this we must learn how to reason, how to prove truths. And in order to avoid such mistakes as that of my friend, who thought he had proved the false proposition of which I have been speaking, we must learn to reason correctly.

25. When we put the corners of a paper triangle together to make a straight line, we may say, Perhaps there is some slight error here, too small to be detected by measurement. How then shall we prove that there is no such error in a perfect geometrical triangle?

26. The first thought that occurs to us will be, that if any straight line be drawn through one vertex of a triangle, as D E is drawn through the point A, without passing through the triangle, the three angles on one side of the line, about the point A are equal to two right angles, and if the sum of the three angles of the triangle is equal to two right angles, it must be equal to that of the three about the point A.




27. But as the central angle at A is already an angle of the triangle, it follows that the other two angles must be equal in their sum to the sum of the angles B and C.

28. Now this will be true in whatever direction the line D E is drawn, only provided it does not pass through the triangle. Let us then imagine it to pass in such a direction as to make the angle B A E equal to the angle A B C and it will only be necessary to prove that D A C is then equal to A C B. For if D A C is equal to A C B, then since we suppose B A E equal to A B C, and B A C is one of the angles of the triangle, we shall have the three angles about A equal to the three angles of the triangle, and as the three angles about A are equal in their sum to two right angles, the three angles of the triangle will be equal to two right angles, which is what we wish to prove.

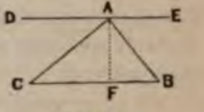


Fig. B.

29. But to say that D A C is equal to A C B is equivalent to saying that A D and C B differ equally in their direction from the direction of A C, and since A C is a straight line and its direction from A is opposite to its direction from C, this is equivalent to saying that A D and C B go in opposite directions.

30. All that we have now to prove is, that the line A D or E D goes in the same direction as the line B C. But this needs no proof, because we have already supposed that E A makes the same angle with A B that B C does; and as B C is a straight line, the direction of E A and C B must be opposite. But as E A is part of the same straight line with A D, it has the same direction as A D. The proof is now complete.

31. And this mode of proof does not depend at all upon the particular shape of the triangle. We have made no supposition concerning the shape of A B C, except that it should be a triangle. We have, therefore, proved that the sum of the three angles of any triangle is equivalent to two right angles.

32. Thus we have analyzed the proposition that the sum of the three angles of a triangle is equivalent to two right angles, and found that it resolved itself at last into saying that two lines making equal angles on opposite sides at the end of a straight line must point in opposite directions, a proposition which is easily shown to be true.

33. But this mode of analyzing is very tedious when stated in words. A geometer usually does not state it; he passes through it very rapidly in his own mind, and then restates the process carefully in an inverted order, as follows in articles 34, 35, and 36.

34. When one straight line crosses another, the opposite or vertical angles are equal. For since each line has but one direction, the difference of direction on one side of the vertex must be the same as on the other side.

35. When one straight line crosses two parallel straight lines, the alternate internal angles are equal, or in the figure (Fig. A.) A G E is equal to D H F. For D H F is equal to B G F, having its sides pointing in the same direction as those of B G F, and A G E is equal to B G F by article 34.

36. Through the vertex of any triangle, as through A (Fig. B.), draw a straight line D E parallel to the opposite side B C. Then E A B will be equal to its alternate internal angle A B C, and for the same reason D A C will be equal to A C B. So that the three angles of

the triangle will be equal to the three angles about the point A, and their sum is plainly two right angles.

37. The mode of proving that the sum of the three angles of a triangle is equal to two right angles, by cutting a piece of card, is called experimental proof. It is of very little use in mathematics, but of great use in the study of physics, especially in mechanics and chemistry.

38. The mode of proof used in articles 26–31 is called, by metaphysicians and by writers on Arithmetic, *analysis*. But as geometers, in their writings, almost never use this method, they have no name for it; and when they speak of analysis or of analytical methods they usually refer to something else of a very different character.

39. The mode of proof in articles 34–36, called by metaphysicians, synthesis, by geometers, demonstration or deduction, is that usually employed in stating geometrical results. This mode is chiefly applicable to mathematics, and must be used with very great caution in reasoning upon other subjects.

40. A proposition which we wish to prove may be compared to a mountain peak which we wish to show is accessible from the highway. The method of articles 26–31 may be compared to taking a flight by a balloon to the top of the peak and then finding a path down to the highway; while the method of articles 34–36 may be compared to the direct ascent of the mountain. In either case we show that the peak is accessible; because we actually pass over all the steps of a connected pathway between the road and the mountain top.

Thus in geometrical demonstration we pass through every step connecting the simplest self-evident truths with the highest deductions of the science; while in the process which writers on Arithmetic call analysis, we pass over every step from those truths down to the simplest. In either case we prove that the higher truth really stands on the same basis as the simpler, and must, therefore, be true.

*See page three of Cover.*

## Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the January number of the Monthly:

WM. E. MERRILL, Cadet, 1st Class, U. S. Military Academy, answered all the questions. (A. E. CHURCH, Prof.).

CHARLES BETTLE, Sophomore Class, Haverford College, Pa., answered all but IV. and V. (M. C. STEVENS, Prof.).

Prof. STEVENS also communicated a very elegant solution of problem IV. by WM. G. RHOADS, Esq., graduate of Haverford College, Class of 1858.

DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y., answered problems III. and V.

W. F. OSBORNE, Sophomore Class, Wesleyan University, Middletown, Ct., answered all but V.

ASHER B. EVANS, Junior Class, Madison University, Hamilton, N. Y., answered all the questions.

We shall publish the report of the Judges, with the Prize Solutions, in the next number.

THE

# MATHEMATICAL MONTHLY.

Vol. I....MAY, 1859....No. VIII.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

If $x$ be the distance of the eye from the centre of a sphere, of which the radius is $r$, prove that the visible part of its surface is to the invisible as $x - r : x + r$.

### II.

Transpose the series

$$1 + 8 + 19 + 34 + 53 + 76 + \&c.,$$

so as to find the sum of $n$ terms by means of the usual formula for summing the squares of the natural numbers.

### III.

If $a, b, c$ are the sides of a spherical triangle, and $A, B, C$ the opposite angles, prove that

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a.$$

### IV.

If, on the sides of a given plane triangle, equilateral triangles be constructed, prove that the triangle formed by joining the centres of these equilateral triangles will also be equilateral; also prove that the straight lines joining the vertices of the equilateral tri-

angles and the opposite angles of the given triangle are equal, and all intersect in the same point.

V.

If an angle ($A$), and the sum of the squares of the sides of a plane triangle, be given ($= 8a^2$), prove that the curve which continually bisects the side opposite to $A$ is an ellipse, and determine the numerical values of its principal diameters when $A = 60°$, and $a = 10$.

The solution of these problems must be received by the first of July, 1859.

## REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. IV., Vol. I.

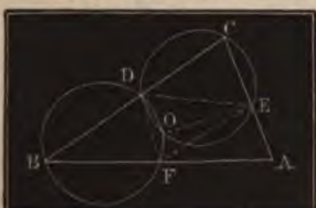
The first Prize is awarded to William E. Merrill, Cadet, First Class United States Military Academy, West Point, N. Y.

The second Prize is awarded to W. F. Osborne, Sophomore Class, Wesleyan University, Middletown, Ct.

### Prize Solution of Problem I.

"If the triangle $DEF$ be inscribed in the triangle $ABC$, the circumferences of the circles circumscribed about the three triangles $AEF$, $BFD$, $CDE$, will pass through the same point."

Let the circles described about the triangles $BFD$ and $DCE$




intersect at the point $O$. Connect $O$ with the vertices $D, E, F$ of the inscribed triangle. Since the angle $FOD$ is the supplement of $B$, and the angle $EOD$ is the supplement of $C$, $\therefore$ the angle $FOE = B + C$. Adding $A$ to both sides, we have $FOE + A = B + C + A = 180°$. $\therefore$ $O$ is on the circumference of the circle circumscribing the triangle $AEF$. This solution is by W. F. Osborne.

### Prize Solution of Problem II.

"Given the base of a spherical triangle, and the ratio of the tangents of the angles at the base; to find the locus of the vertex." — Communicated by George Eastwood, Esq.

Let $A B C$ be the triangle. Refer the vertex $C$ to the base by the perpendicular $C D$. Denote the base by $a$, and $A D$ by $x$; $D B$ will be denoted by $a - x$. From the triangles $A D C$ and $C D B$, Napier's Circular Parts give

$$\sin x = \tan C D \cot A, \quad \sin (a - x) = \tan C D \cot B.$$

$$\therefore \frac{\sin (a - x)}{\sin x} = \frac{\tan C D \cot B}{\tan C D \cot A} = \frac{\tan A}{\tan B} = m = \text{given ratio.}$$

$$\therefore \; m \sin x = \sin (a - x)$$

$$= \sin a \cos x - \cos a \sin x,$$

$$\therefore \sin x (m + \cos a) = \sin a \cos x,$$

$$\therefore \sin^2 x (m + \cos a)^2 = \sin^2 a \cos^2 x = \sin^2 a (1 - \sin^2 x),$$

$$\therefore \sin^2 x (m^2 + 2 m \cos a + 1) = \sin^2 a,$$

$$\therefore \qquad \sin x = \frac{\pm \sin a}{\sqrt{(m^2 + 2 m \cos a + 1)}} = \text{a constant.}$$

Hence $A D$ is constant for all positions of the vertex $C$, and the required locus is therefore a great circle perpendicular to the base. If $m = 1$, $x = \frac{1}{2} a$; that is, the base is bisected by the locus. This solution is by William E. Merrill.

### Prize Solution of Problem III.

"If in any triangle a line be drawn from the vertex of either angle to the opposite side, bisecting the angle, prove that the product of this line and the secant of half the bisected angle equals a harmonic mean between the two sides containing the bisected angle." — Communicated by Prof. J. M. VanVleck.)

Let the angle $A B C$, which is bisected by $B D$, equal $2 \theta$. Then the area of the triangle $A B D = \frac{1}{2} A B \times B D \sin \theta$; the area of $C B D = \frac{1}{2} B C \times B D \sin \theta$; the area of $A B C = \frac{1}{2} A B \times B C \sin 2 \theta$.

$$\therefore \tfrac{1}{2} B C \times B D \sin \theta + \tfrac{1}{2} A B \times B D \sin \theta = \tfrac{1}{2} A B \times B C \sin 2 \theta,$$

$$\therefore\ (BC+AB)\,BD\sin\theta = AB\times BC\sin 2\theta$$
$$= 2\,AB\times BC\sin\theta\cos\theta,$$

$$\therefore\qquad BD\sec\theta = \frac{2\,AB\times BC}{AB+BC}.$$

This solution is by ASHER B. EVANS. The problem was solved in the same manner by W. L. OSBORNE.

### PRIZE SOLUTION OF PROBLEM IV.

"If $A, B, C$, and $D$ be any four points in the same plane, so situated that four circles, $ABC$, $ABD$, $ACD$, and $BCD$, can be drawn through them three and three, prove that the circumferences of any two of these circles will intersect at the same angle as the circumferences of the remaining two." — Communicated by Prof. H. A. NEWTON.

Let $A, B, C, D$ be the four points, joined by straight lines, two


and two. Since each of these lines is a common chord to two of the circles, and since the angle formed by a tangent and a chord is equal to the inscribed angle measured by the same arc, we have $ABD = DAE$ and $ACD = DAF$. $\therefore\ ABD - ACD = DAE - DAF = FAE =$ the angle made by two of the circumferences.

Again, $BDC = BCG$, and $BAC = BCH$.

$\therefore\ BDC - BAC = BCG - BCH = HCG$.

But $AOD = BAC + ABD = ACD + BDC$.

$\therefore\ ABD - ACD = BDC - BAC = FAE = HCG$.

In precisely the same way the proposition may be proved for any other circumferences two and two.

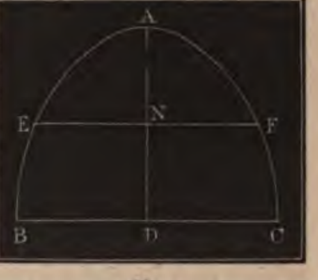
It is evident that either point, as $D$, may fall within the triangle formed by joining the other three points; but it will not be difficult

to modify the figure, and prove the proposition for this case. This solution is by W. F. OSBORNE.

PRIZE SOLUTION OF PROBLEM V.

"A paraboloid of given dimensions but unknown specific gravity is immersed in common water, until its summit coincides with the surface of the fluid. The pressure from above being removed, the body ascends by the force of the water until its base coincides with the fluid's surface, and then descends, and so on. Find from this circumstance the specific gravity of the body." — Communicated by GEORGE EASTWOOD, Esq.

Let $B\,A\,C$ represent a vertical section of the paraboloid through its axis $A\,D$. Let $A\,N = x$, $E\,N = y$, $A\,D = a$, $B\,D = b$. Then from a property of the parabola $b^2 : y^2 :: a : x$. Whence $y^2 = \frac{b^2 x}{a}$. Let $E\,N\,F$ represent the surface of the water at any instant. Then the amount of water displaced equals the solidity of the frustrum




$$E\,B\,C\,F = A\,B\,C - A\,E\,F = \tfrac{1}{2}\pi\, b^2 a - \tfrac{1}{2}\pi\, y^2 x = \frac{\pi b^2}{2a}(a^2 - x^2).$$

Let $g =$ the specific gravity of the paraboloid, then its weight, water being the standard unit, $= \frac{\pi}{2} b^2 a g$. Then

$\frac{\pi b^2}{2a}(a^2 - x^2) - \frac{\pi}{2} b^2 a g = \frac{\pi b^2}{2a}(a^2 - x^2 - a^2 g) = F =$ the force tending to elevate the paraboloid at any instant. If the summit $A$ was at the surface of the water when the ascent commenced, the space $s = A\,N = x$. Let $v =$ the velocity due to the space $s$; then, since the accelerating force is variable, we have $F d s = m v d v$; $m$ denoting the mass of the paraboloid.

Substituting for $F$ and $s$ their values as found above, we have

$$\frac{\pi b^2}{2a}(a^2 - x^2 - a^2 g)\, d x = m\, v\, d v.$$

Integrating $$\frac{\pi b^2}{2a}(a^2 - a^2 g)\, x - \frac{\pi b^2}{2a} \cdot \frac{x^3}{3} = \tfrac{1}{2} m v^2.$$

As $v=0$ when $x=a$, we have

$$\frac{\pi b^2}{2a}(a^2-a^2 g)\,a-\frac{\pi b^2}{2a}\cdot\frac{a^3}{3}=0.$$

Whence, from this equation, $g=\frac{2}{3}$ of the specific gravity of water. This solution is by Asher B. Evans.

Joseph Winlock.
Chauncey Wright.
Truman Henry Safford.

## THE CONIC SECTION COMPASSES.

By Joseph P. Frizell, Lowell, Mass.

The frequency with which the ellipse, parabola, and hyperbola occur in architecture and mechanics, the extensive applications of which they are capable in the solution and illustration of algebraical and geometrical questions, and their many interesting properties, which can be fully appreciated only when they are correctly drawn, render an instrument which describes them with readiness and precision, a valuable addition to our present stock of mathematical apparatus.

The instrument represented in the accompanying drawing is designed to effect this object, which it does by virtue of the well-known property which gives to these curves the name of conic sections. I shall first describe the instrument, and, next, indicate the theoretical considerations on which its operation depends. $M$ is a heavy rectangular block of metal with two beveled edges, the weight of which serves to keep the instrument in its position. At right angles to the plane of its base is fixed the vertical pillar $S$, which is graduated to inches and twentieths upon its face $fg$. Upon this pillar slides vertically the disc $F$, capable of being fixed at any height, by means of a clamp screw, the nonius or index

whereby it is thus adjusted coinciding, in vertical height, with the point $c$. On the face of the disc, next the pillar, is graduated an arc of ninety degrees, the centre of which is at $c$, and its zero in a horizontal line passing through $c$. Upon the front face of this disc

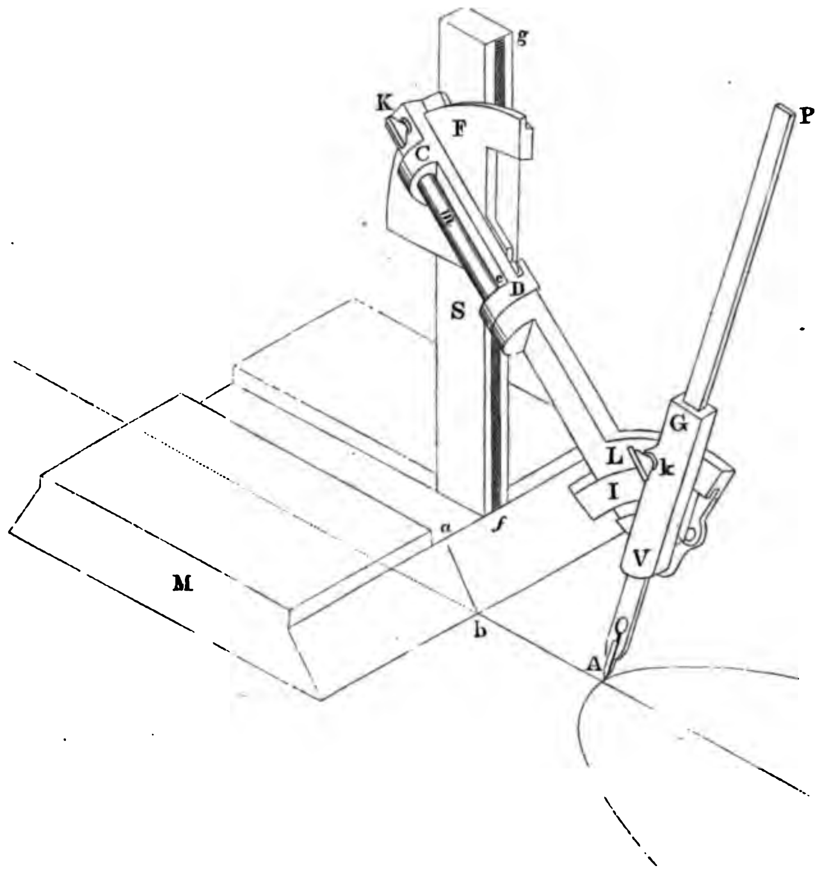



Fig. 1.

is fixed the guide $CD$, which radiates about the centre $c$, and is adjustable upon the disc at any angle of the quadrant, by means of an index, vernier, and clamp screw at $R$. Two lugs, formed upon this guide at $C$ and $D$, embrace the axis or journal, $m$, which revolves in the lugs, and is capable of no other motion. This axis is

extended beyond the guide $CD$; and upon its extension is formed another small disc, $L$. To this disc is jointed another guide, $GV$, radiating about the point $V$, which is in a straight line passing through $c$ and the index at $K$. The guide $GV$ may be adjusted, at any angle, to the axis $KV$ by a clamp screw $k$ and a limb $I$ fitting into a corresponding groove in the disc $L$; the united length of the limb and groove being, if necessary, available, thereby allowing of any inclination $GVK$ from near zero to ninety degrees. A perfectly straight rôd or ruler, $AP$, carrying a pencil at its extremity $A$, is fitted to slide accurately, but with as little resistance as possible, through the guide $GV$; its central line passing through the point $V$. It must be observed, that by the point $V$ is meant the point where the short axis, around which $GV$ radiates, is intersected by the axis of the cylindrical journal $m$ produced; and that by the point $c$ is meant the point where the axis around which $CD$ radiates is intersected by the axis of $m$. On each edge of the block $M$ is marked a line coinciding with a vertical plane passing through the points $V$ and $c$; $b$ being vertically under $c$.

Suppose, now, the instrument to be standing upon a table or drawing board, the point $A$ of the pencil touching the surface of the paper. If the journal $m$ be made to revolve, carrying around the rod $PA$, this latter will generate the two nappes of a cone having its vertex at $V$. If, during such revolution, the rod $PA$ be slipped through the guide so as to keep the point of the pencil in contact with the paper, it will describe a conic section determined in species, magnitude, and position, by (1) the angle at which the index $K$ is placed upon the disc $F$; (2) the angle made by the generatrix $PA$ with the axis; (3) the position of the point $V$.

I proceed to the theoretical considerations which indicate the method of adjusting the instrument so as to draw any given curve.

Two questions present themselves in this connection: name-

ly, (1) What conditions must a plane satisfy, in order that its intersection with a given cone may be a given curve? (2) What conditions must a cone satisfy, in order that its intersection with a fixed plane may be a given curve? To solve the first: Suppose a cone, in which $v$ represents the angle at the base, to be intersected at a distance $a$ from the vertex (measured on the slant side) by a plane, making an angle $u$ with the base. The general equation of the curve of intersection will be

$$(1) \qquad y^2 = 2\,a\,x\,\cot v\,\sin(v+u) - x^2\left(1 - \frac{\sin^2 u}{\sin^2 v}\right)$$

in which the curve is referred to its vertex and transverse axis.

The general equation of a conic section, referred to its vertex and transverse axis, is

$$(2) \qquad y^2 = 2\,m\,x\,(1+e) - x^2\,(1-e^2)$$

where $m$ is the distance from the focus to the vertex, and $e$ is the eccentricity. Therefore, by the principle of indeterminate coefficients $1 - \frac{\sin^2 u}{\sin^2 v} = 1 - e^2$, and $2\,a\cot v\,\sin(v+u) = 2\,m\,(1+e)$. Therefore, $\sin u = e\sin v$ (3), and $a = \frac{m\,(1+e)}{\cot v\,\sin(v+u)}$ (4), which are the required conditions.

The solution of the second question is contained in the two following propositions.

(1) If a sphere be enveloped by a cone whose surface is intersected by a plane tangent to the sphere, the point of tangency is a focus of the section.

(2) The locus of the vertices of all the right cones whose intersection with a given plane is a given curve, is a conic section, having its vertices at the foci, and its foci at the vertices of the given curve; its plane being perpendicular to the given plane, and intersecting it in the transverse axis. If the given curve be an ellipse, the locus is an hyperbola; if an hyperbola, the locus is an ellipse;

if a parabola, the locus is also a parabola. Both these properties are readily deduced from Equation (3); and, when I began to write this article, I was not aware that the second proposition had ever been stated before. I have since found them both in the *Cambridge Philosophical Transactions*, Vol. III., Part 1, Memoir 8, where they seem to be claimed as original discoveries by PIERCE MORTON. A reference to this authority renders a demonstration unnecessary.

These properties will be more readily understood from an examination of Fig. 2, which is supposed to be an axial section of a cone or system of cones; $AB$ being the projection of a cutting plane, and the circles tangent at $F, F'$, sections of the inscribed spheres.

Fig. 2.

The curve of intersection may be regarded as an ellipse whose vertices are $A, B$, and foci $F, F'$; an hyperbola whose vertices are $A', B'$, and foci $F, F'$; or a parabola whose vertex is $A$ and focus $F$. In the first case, the vertex of the cone is at $V$; in the second at $V'$; in the third at $V''$. By supposing the circle tangent at $F$ to vary in magnitude, it is shown, in the first case, that the locus of $V$ is an hyperbola having its foci at $A, B$, and its vertices at $F, F'$; in the second case, that the locus of $V'$ is an ellipse having its vertices at $F, F'$ and its foci at $A', B'$; in the third case, that the locus of $V''$ is a parabola, having its vertex at $F$ and focus at $A$. It is obvious that the axis of the cone is tangent to the locus at $V\ V'\ V''$, since, in the case of the ellipse and hyperbola, it makes equal angles with lines drawn from $V, V'$ to the foci, and in the case of the parabola it bisects the angle formed by a line to the focus, and a perpendicular upon the directrix.

These properties indicate the method of adjusting the instrument so as to draw any given conic section. From the equation of the given curve, or from the data by which it is given, the equation of the corresponding locus must be found. Any convenient point in this locus being chosen by assuming one of its coördinates, and finding the other from its equation, the angle which a tangent to the locus at that point makes with the transverse axis must be found. This angle must be laid off on the disc $F$. The point $V$ in the instrument must be placed at the point chosen in the locus. The inclination $G\,V\,K$ must then be made such that the point $A$ of the pencil shall touch the vertex of the given curve. The instrument is then ready to draw the curve.

As an example of this operation, let it be required to draw an ellipse whose semi-transverse axis is $A\,O$, and whose semi-focal distance is $F\,O$. We are first to find the angle to be laid off on the disc $F$. This may be done in either of two ways. (1) Let the equation of the locus of $V$ be formed, which will be $A^2 y^2 - B^2 x^2 = A^2 B^2$, in which $A^2 = A\,O^2, B^2 = A\,O^2 - F\,O^2$, and $x, y$ are the general coördinates. In this locus choose any convenient point to be occupied by $V$ (Fig. 1). Designate its coördinates by $x' y'$, $y'$ being any assumed distance $V n$, and $x' = \sqrt{A^2 + \frac{A^2}{B^2} y'^2}$. The equation of a tangent to the locus at $V$ is $A^2 y y' - B^2 x x' = - A^2 B^2$, in which, if we make $y = 0$, we have $x = \frac{A^2}{x'}$. Make $O\,P = \frac{A^2}{x'}$, and through $P$ draw $V\,P$. The angle $V\,P\,n$ is the angle required. (2) A very much simpler, though perhaps slightly less accurate, method of finding this angle, is by a geometrical construction indicated in the figure; thus, erect the perpendicular $F t$, upon which with any convenient radius $C\,F$ draw a circle. From $A$ and $B$ draw tangents to this circle, and from $V$, their point of meeting, draw through $C$ a line meeting, $A\,B$ in $P$. $V\,P\,n$ is the a[illegible] required, as

before. The angle thus found must be laid off upon the disc $F$, and the axis clamped in that position by the thumbscrew $K$. Upon $PV$ produced, take $Vc = VC$ (Fig. 1), and draw $cb$ perpendicular to $AB$. Next, place the edge $b$ of the instrument at $b$ (Fig. 2), taking care to make the line $ab$, and its corresponding line on the opposite edge of the block $M$, coincide with the prolongation of the transverse axis. Elevate the disc $F$ till its index is at a distance $cb$ above the paper, in which position let it be clamped. Make the inclination of the generatrix to the axis such that $A$ may touch the vertex of the curve, which can then be drawn.

If, from the great eccentricity of the ellipse, the pencil meets the paper near the opposite vertex, at too great an inclination to produce a definite line, one half the curve can be drawn, and then the instrument must be removed to a corresponding position on the opposite side of the centre to draw the other half.

In a manner entirely similar to that described for the ellipse, the instrument may be adjusted to draw either of the other conic sections.

## ON CONTACT, CENTRES OF SIMILITUDE, AND RADICAL AXES.*

By Matthew Collins, B. A., Dublin, Ireland.

1. The straight line joining the ends of any two parallel radii of two given circles passes through a fixed point in the line joining their centres; namely, the point where the distance of the centres is cut (externally or internally) in the ratio of the radii.

Let $O$ and $O'$ be the centres of two given circles, $OC$ and $O'C'$ any pair of parallel radii, and $P$ the point of intersection of $CC'$

* See Note at the end of this article.

and $O\,O'$; then as $O\,C$ is parallel to $O'\,C'$ $\therefore$ the triangles $P\,O\,C$, $P\,O'\,C'$ are similar; therefore $P\,O : P\,O' = O\,C : O'\,C'$, and therefore constant; and as points $O$ and $O'$ are fixed, therefore $P$ must be so too.

But if $O'\,C''$ be parallel and contrary to $O\,C$, then $O\,O'$ will obviously be cut internally by $C\,C''$ at $P'$, in the ratio of the radii, so that $P'$ will be fixed as well as $P$, and $O\,O'$ will therefore be cut harmonically at $P$ and $P'$, which are called the external and internal centres of similitude of the two given circles.

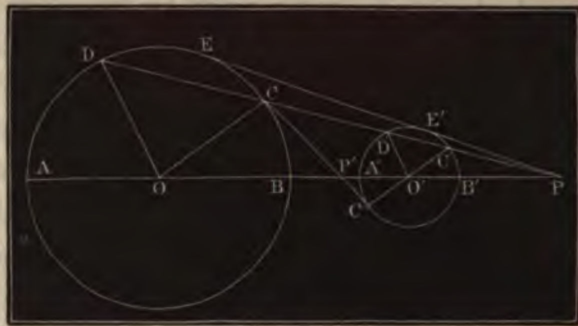


Fig. 1.

Corollary. Hence the point of contact of two circles is obviously their external or internal centre of similitude, according as the circles touch each other internally or externally; and a common tangent $E\,E'$ to two circles always passes through their centre of similitude, since the radii $O\,E$ and $O'\,E'$, passing through the points of contact $E$ and $E'$, are both perpendicular to the common tangent, and therefore parallel to each other.

2. If $O\,C$ be paralled to $O'\,C'$, then $O\,D$ must be parallel to $O'\,D'$, $D$ and $D'$ being the points where $C\,C'$ cuts the circles. Again, the angle $O\,D\,C = O\,C\,D = O'\,C'\,D' = O'\,D'\,C'$, and therefore $O\,D$ is parallel to $O'\,D'$; that is, $D$ and $D'$ will correspond if $C$ and $C'$ correspond.

3. Conversely, if from the centre of similitude $P$ of two circles any line be drawn cutting them in $C', D', C, D$, so that $C$ corresponds to $C'$, then $P\,C \times P\,D' = P\,C' \times P\,D$ will be constant; and therefore $P\,B \times P\,A' = P\,A \times P\,B' = P\,E \times P\,E'$. For as $C$ corresponds to $C'$, $O\,C$ is parallel to $O'\,C'$, therefore $P\,C : P\,C'$ is con-

stant ($= O\,C : O'\,C'$), and as $P\,C' \times P\,D'$ is constant, therefore also $P\,C \times P\,D'$ is constant.

Obs. If the circles touch at $B$, $A'$ will coincide with $B$, and the constant value of $P\,C \times P\,D'$ will then be $P\,B^2$.

4. By supposing one of the two given circles to become infinite, it follows that the centre of similitude of a straight line and circle is that point $P$ on the circumference at the greatest (or least) distance from the given straight line and, moreover, that if through this point $P$ any straight line be drawn cutting the given straight line and the circle in $C$ and $C'$, then $P\,C \times P\,C'$ will be constant, as is easily demonstrated otherwise, directly.

5. The radical axis of two circles is the locus of a point from which the tangents to the two given circles are equal to each other and is a straight line. For if the tangent $P\,C =$ the tangent $P\,C'$, $O$ and $O'$ being the centres, $A$ the middle point of $O\,O'$, and $P\,B$ perpendicular to $O\,O'$, then $2\,O\,O' \times A\,B = P\,O^2 - P\,O'^2 \therefore = O\,C^2 - O'\,C'^2$; and as $O\,C$, $O'\,C'$, and $O\,O'$ are given, $\therefore AB$ is given, and therefore the point $B$ is fixed, and so the locus of $P$ is a straight line perpendicular to $O\,O'$ at $B$, when $O\,O'$ is divided, so that $O\,B^2 - O'\,B^2 = O\,C^2 - O'\,C'^2$.

Fig. 2.

Corollary. The radical axis of two circles that cut each other is their common chord, and the radical axis of two circles that touch each other is their common tangent at their point of contact, as is directly evident.

6. If a variable circle touch two fixed circles, the line joining the points of contact will pass through their centre of similitude, and the tangent from this centre to the variable circle will be constant.

Let the variable circle $O''$ touch the two fixed circles $O$ and $O'$ at $C$ and $D'$, and let $C D'$ produced cut circle $O'$ and the line $O' O$ in $C'$ and $P$. Now $O O''$ and $O' O''$ pass through $C$ and $D'$, and angle $O' C' D' =$ angle $O' D' C'$ $\therefore =$ angle $O'' D' C$ $\therefore =$ angle $O'' C D'$, and therefore $O' C'$ is parallel to $O C O''$; and therefore by Arts. 1 and 3, $C D'$ passes through $P$, and $P T^2 = P C \times P D'$ is constant, $P$ being here the external centre of similitude of the two fixed circles $O$ and $O'$, and $P T$ being a tangent to the variable circle $O''$.

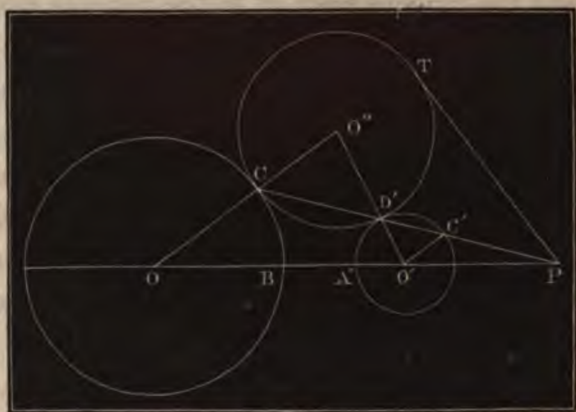


Fig. 3.

Cor. 1. Hence if a variable circle $O''$ touch two fixed circles $O$ and $O'$, it will also cut orthogonally another fixed circle whose centre is $P$ and rad. $= P T$; and therefore conversely, if a variable circle touch a fixed circle, and cut another fixed circle orthogonally, it must also touch another fixed circle.

Note. If the variable circle touched one of the two given circles externally, and the other internally, then the point $P'$ through which the chord of contact always passes would obviously be the internal centre of similitude of the two given circles; but in all other cases $P$ will be the centre of similitude.

Cor. 2. If the two given circles touch each other externally at $B$, so that $A'$ coincides with $B$, then $P$ must be the external centre of similitude, and the tangent $P T$ to the variable circle will be equal to $P B$, since by Art. 3, Obs., $P B^2$ is then equal to $P C \times P D' = P T^2$. But if the two given circles touched each other internally at $B$, then by the foregoing Note, the line $C D'$ joining the points

of contact will pass through their internal centre of similitude $P'$, and the tangent from $P'$ to the variable circle will then too be equal $P'B$.

Cor. 3. If one of the given circles becomes very large, its circumference becomes nearly straight, as in Art. 4; and hence if a variable circle touch a given straight line and a given circle, the chord of contact will pass through a fixed point on the circumference of the given circle, and the tangent from this point to the variable circle will be constant, which could be easily otherwise demonstrated directly.

Cor. 4. If another circle $O'''$ also touch the two circles $O$ and $O'$ at $c$ and $d'$, then by this Art. 6, $cd'$ must also pass through $P$, the centre of similitude of $O$ and $O'$, and the tangent from $P$ to circle $O'''$ will be equal $PT$ the tangent from $P$ to $O''$, and therefore, by Art. 5, $P$ must be a point on the radical axis of $O''$ and $O'''$. Hence this theorem, namely, *If each of two circles touch* (*in the same way*) *another pair of circles, the centre of similitude of either pair lies upon the radical axis of the other pair*; if each of the pair of circles $O'', O'''$ touches one circle of the other pair ($O$, $O'$) externally, and the other circle internally, then (by Note, Art. 6) $P$ would be the internal centre of similitude of $O$ and $O'$; but in all other cases $P$ will be their external centre of similitude.

Cor. 5. If we conceive the circle $O'''$ to remain fixed, as well as the two given circles $O$ and $O'$, and $O''$ alone to vary, then, by the preceding Cor. 4, the centre of similitude $Q$ of $O''$ and $O'''$ must lie upon the radical axis $AB$ of $O$ and $O'$; let $d$ and $x$ be the distances of $AB$ from the centres of $O'''$ and $O''$; then by Art. 1, rad. of $O''$ : rad. of $O''' = QO'' : QO'''$, therefore by similar angles equal $x : d$; or rad. of $O'' : x =$ rad. of $O''' : d$, and hence we have the following most useful and important theorem; namely, *If a variable circle touch two fixed circles, its radius varies as the distance of its centre from the radical*

*axis of the given circles;* and therefore conversely, if a variable circle touch a given circle, and cut a given straight line at a given angle, or, more generally, if its radius vary as the distance of its centre from a given straight line, it shall also touch another given circle.

I shall here give a few remarkable applications of the foregoing useful theorem.

1st. Let $O\,B\,D$ be a quadrant, and $C$ the centre of circle $L\,E\,G$ inscribed in it, and let $n$ be the centre of the circle $P\,F\,m$ touching the two former and the radius $O\,B$; then $P\,O = 7$ times $P\,n$.

For the circle $L'\,E\,G'$, inscribed in the adjacent quadrant, $O\,B\,D'$ will obviously be equal to circle $C$, and touch it in $E$, and $O\,C'$ will be parallel to the tangent $L\,K$, as both are perpendicular to $O\,L$. Complete the rectangle $H\,L\,K\,n$, and produce $H\,n$ to $M$ and $N$; then as (by Art. 5, Cor.) $L\,K$ is the radical axis of

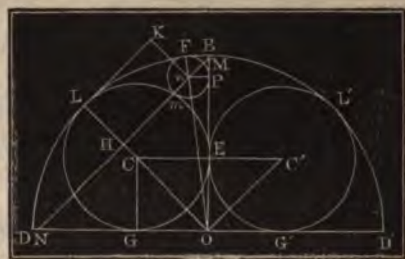


Fig. 4.

circles $O$ and $C$, which are both touched (in the same sort of way) by circles $n$ and $C'$; therefore, by the foregoing theorem, Cor. 5, $n\,K : n\,P = O\,L : C'\,E$; and as $O\,L = O\,C + C\,E$ is the sum of the side and diagonal of a square, $G\,E$, whose side $C\,E = C'\,E$; therefore $n\,K$ must equal the side and diagonal of a square described on $n\,P$, and therefore equal to $n\,P + n\,M$, since $M\,P\,n$, $O\,H\,N$, $O\,C\,C'$ are obviously isosceles right-angled triangles. Now, as $O\,F = O\,L$, therefore $O\,n - O\,H = H\,L - n\,F = n\,K - n\,P$, and therefore $= n\,M = H\,M - H\,n$; that is, equals $O\,H - H\,n$. Thus the three sides of the right-angled triangle $n\,H\,O$ are in arithmetical progression, and therefore they are to each other as 3, 4, and 5. But $n\,N = n\,H + H\,O$ and $n\,M = H\,O - H\,n$; hence, then, $n\,N : n\,M = 4 + 3 : 4 - 3$, that is, equal $7 : 1$; but $P\,O : P\,n = P\,O : P\,M$; and therefore by similar triangles $= n\,N : n\,M = 7 : 1$.

Cor. The point of contact $m$ is four times as far from $OD$ as from $OB$. For if $Cmn$ produced, meet $OB$ in $R$, then by similar triangles $RCE$, $RnP$, we get $RC:Rn=CE:nP$; that is, equals $Cm:mn$; therefore $RC$ is cut harmonically at $m$ and $n$, and so $RC$, $Rm$, $Rn$ are in harmonic progression, and therefore $CE$, $mm'$, and $nP$, which are proportional to them, are also in harmonic progression, $mm'$ being perpendicular to $OB$; and therefore their reciprocals, namely, $\frac{OR}{CE}$, $\frac{OR}{mm'}$ and $\frac{OR}{nP}$ are in arithmetical progression; and of course they will remain so still when diminished by the equals $\frac{RE}{CE}=\frac{Rm'}{mm'}=\frac{RP}{nP}$; that is, $\frac{OE}{CE}$, $\frac{Om'}{mm'}$ and $\frac{OP}{nP}$ are in arithmetical progression, and as $\frac{OE}{CE}$ is obviously equal 1, and $\frac{OP}{nP}$ was proved equal 7, therefore $\frac{Om'}{mm'}$ must equal 4.

The foregoing demonstration is easier than and preferable to that inserted by me in "*The Educational Times*" for June, 1857. The celebrated Thomas Simpson first discovered this theorem ($OP=7\times Pn$), and gave, in page 284 of his Algebra, an algebraical demonstration of it which is excessively complicated with surd reductions.

2d. Let circles $O$ and $O'$ touch the ordinate $CD$, and the semicircles $ADB$, $AdC$, and $Bd'C$. I say circle $O=$ circle $O'$. For the tangent $AF$ is the radical axis of circles $ADB$ and $AdC$, both of which are touched by the circles $O$ and $Bd'C$, whose centre is $G$; draw $EOF$ parallel to $AB$, through the centre $O$ and the point of contact $E$, then by foregoing Cor. 5, $OF:OE=GA:GB$ ∴ $EF:2OE=AB:BC$; and as $EF=AC$ ∴ $\frac{AC\times BC}{AB}=2OE$; that is, the diameter of

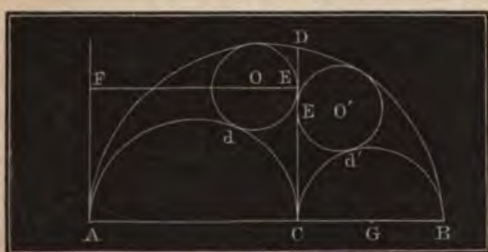


Fig. 5.

circle $O$ is one half a harmonic mean between $A\,C$ and $B\,C$; and the same value would obviously be found for the diameter of circle $O'$.

NOTE. It is easy to prove that $C\,D^2 = C\,E^2 + C\,E'^2$, and $C\,E \times C\,E' = C\,D \times$ the diameter of $O$ or $O'$.

3d. Again, let $O\,B\,L\,D$ be a quadrant, $O\,O'$ its circumscribed square, $D\,F\,G\,O$ and $B\,H\,G\,E$ semicircles whose centres are $C$ and $P$ touching each other in $G$, and $A$ the centre of a circle touching the three former; then rad. $P\,B = \frac{1}{3}\,O\,B$, and rad. $A\,L = \frac{1}{6}\,O\,B$, and $G$ will be twice as far from $O\,D$ as from $O\,B$, and $A\,C\,O\,P$ will be a rectangle. For as $D\,O'$ is the radical axis of the circles whose centres

Fig. 6.

are $O$ and $C$, which are both touched by the circles $P$ and $C'$ ($O\,C'$ being taken equal and opposite to $O\,C$), and as $C'\,O = \frac{1}{3}\,C'\,D$ $\therefore$ by Cor. 5, the rad. of $P = \frac{1}{3}$ of $P$'s distance from $D\,O'$; that is, $P\,B = \frac{1}{3}\,O\,D = \frac{1}{3}\,O\,B$; and rad. of $A = \frac{1}{3}$ of $A$'s distance from $D\,O'$; hence $O\,B$ is trisected at $P$ and $E$. Now, the distances of $G$ from $O\,D$ and $O\,B$ are obviously equal the distances of $C\,G\,P$ from $O$ and $E$; and therefore the former is double the latter, since $P\,O = 2\,P\,E$. Again: as $B\,O'$ is the radical axis of circles $O$ and $P$, which are both touched by the circles $C$ and $A$, and as the radius of $C$ is obviously $= \frac{1}{2}$ of $C$'s distance from $B\,O'$; therefore, by Cor. 5, the radius of $A = \frac{1}{2}$ of $A$'s distance from $B\,O'$. Now draw $c\,c'$ and $p\,p'$ through $A$ parallel to the sides of the square, and therefore equal to them. Therefore $A\,c' = 2\,A\,L$, and $A\,p' = 3\,A\,L$, and therefore $O\,L - A\,L$, $O\,B - A\,c'$, and $O\,D - A\,p'$; that is, $O\,A$, $A\,c$, $c\,O$ are in arithmetical progression, and, as they form a right-angled triangle, therefore they are to each other as 3, 4, 5. Hence if $R =$ radius of $O$, and $r =$ rad. of $A$, $R - 2\,r : R - 3\,r = 4 : 3$, which gives $R = 6\,r$;

and so $A\,c$ (or $O\,p$) $= R - 2\,r \therefore = \frac{2}{3}\,R = O\,P$, and $A\,p$ (or $O\,c$) $= R - 3\,r \therefore = \frac{1}{2}\,R = O\,C$; so the points $c$ and $p$ coincide with $C$ and $P$, and therefore $A\,C\,O\,P$ is a rectangle.

4th. Again: if the circle whose centre is $A$ touch the quadrant $B\,L\,D$, and the semicircles described upon its limiting radii $O\,B$, $O\,D$, then

rad. of $A$ : rad. of $B\,L\,D = \sqrt{2} - 1 : 3\sqrt{2} - 1 = 1 : 5 + 2\sqrt{2}$.

Since the radical axis of each pair of three circles meets in one point, therefore the chord $O\,G$ must pass through the opposite corner of the circumscribed square $O\,O'$; and it is evident that the centre $A$ and the point of contact $L$ will lie on $O\,G\,O'$. Complete the square $A\,C\,O'\,P$, and for the reasons assigned in the foregoing (3d), $A\,C$ or $A\,P = 3$ times rad. $A\,L$.

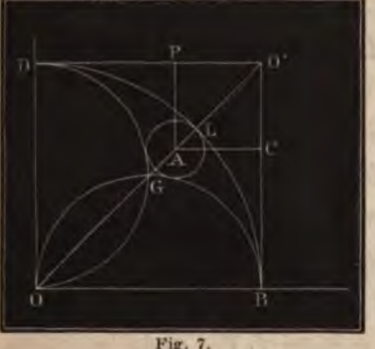


Fig. 7.

$$\therefore A\,O' = A\,P\sqrt{2} \therefore = 3\,A\,L\sqrt{2},$$

and $\therefore A\,L\,(3\sqrt{2} - 1) = L\,O' = O\,O' - O\,L = O\,B\,(\sqrt{2} - 1)$; hence $A\,L : O\,B = \sqrt{2} - 1 : 3\sqrt{2} - 1$; or $\therefore O\,B = A\,L\,(5 + 2\sqrt{2})$. See Ques. 351 of Colenso's Trigonometry.

5th. If the semicircles on the diameters $A\,B$, $A\,C$, touching each other at $A$, be both touched by the circles whose centres are $O$ and $O'$, which touch each other at $T$; demit $O\,F$ and $O'\,G$ perpendiculars on $A\,C\,B$; then if $O\,F = n$ times diameter of $O$, $O'\,G$ will be equal $(n + 1)$ times diameter of $O'$, $O'$ being nearer to $A$ than $O$ is. For, by

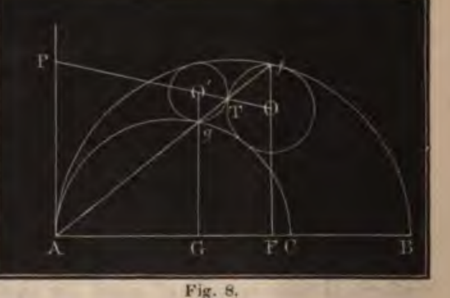


Fig. 8.

Art. 6, Cor. 4, the centre of similitude $P$ of $O$ and $O'$ will lie upon the radical axis of the semicircles on $A\,B$, $A\,C$, which is their common

tangent $AP$ at their point of contact $A$ (Art. 5, Cor.); and as (by Art. 6, Cor. 2), $PA = PT$, $\therefore$ by similar triangles $O'T = O'g$ and $Of = OT$, $f$ and $g$ being the points where $AF$ cuts $OF$ and $O'G$; $f$ and $g$ must therefore be upon the peripheries of circles $O$ and $O'$. Now, by similar triangles, $Ff : Gg = AF : AG = PO : PO'$, $\therefore$ by Art. $1 = Of : O'g$, and $\therefore Ff + Of : 2\,Of = Gg + O'g : 2\,O'g$; that is, $Of + 2\,Of : 2\,Of = O'G : 2\,O'g$; and as by the hypothesis $OF = n$ times $2\,Of$ $\therefore$ $OF + 2\,Of = (n+1)$ times $2\,Of$; and hence also $O'G = (n+1)$ times $2\,O'g$. The same proof holds true if the two original semicircles touched each other externally at $A$.

Cor. As the semicircle on diameter $BC$ touches the two given semicircles, and as the distance of its centre from $AB$ is zero, or $=0$ times its diameter, therefore, if circle whose centre is $O$, touches these three semicircles, the distance of its centre from $AB$ will be equal its diameter; and thence, if the circle whose centre is $O'$ touch the circle $O$ and any two of the three semicircles, the distance of its centre from $AB$ will be twice its diameter, and if the circle whose centre is $O''$ touch the circle $O'$ and touch the same two of the three semicircles that $O'$ touches, then the distance of $O''$ from $AB$ will be three times diameter of circle $O''$, and so on.

The preceding beautiful theorem, so remarkable for elegance and generality, was known to the ancient geometers under the name of Ἄρβηλος, or, *The Shoemaker's Knife.*

Scholium. When one of the two original semicircles becomes infinite, the truth of the theorem follows at once from the principle, that the common tangent to two circles that touch each other is a mean proportional between their diameters. For, let the circle whose diameter is $\Delta$ and its tangent at $A$ be touched by the circles whose diameters are $d_1, d_2, d_3$, which touch each other consecutively, $B$, $C$, and $D$ being their points of contact with their com-

mon tangent, then $AB^2 = \varDelta d_1$, $AC^2 = \varDelta d_2$; and $BC^2 = d_1 d_2$; therefore $\frac{AB^2 \times BC^2}{AC^2} = d_1^2$ and $\frac{AC^2 \times BC^2}{AB^2} = d_2^2, \therefore \frac{AB}{d_1} = \frac{AC}{BC}$ and $\frac{AC}{d_2} = \frac{AB}{BC}$ and as $\frac{AC}{BC} = \frac{AB + BC}{BC} = \frac{AB}{BC} + 1$, hence if $\frac{AC}{d_2} = n$, $\frac{AB}{d_2}$ will be $= n + 1$, which proves the theorem in this particular case.

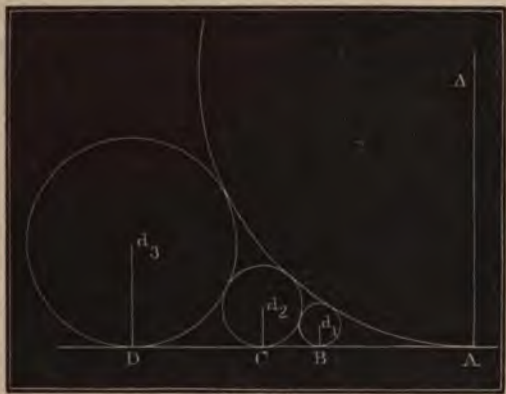


Fig. 9.

Remark. Since also $AD^2 = \varDelta d_3$, and $CD^2 = d_2 d_3$, $\therefore AB^2 \times CD^2 = \varDelta d_1 d_2 d_3$, $\therefore = AD^2 \times BC^2$, and therefore $AB \times CD = AD \times BC$, so that $AD$ is cut harmonically at $B$ and $C$, and therefore $AC \times BD = 2\,AB \times CD$, and therefore $BD^2 = \frac{4\,AB^2 \times CD^2}{AC^2}$, that is $= \frac{4\,\varDelta d_1 d_2 d_3}{\varDelta d_2} = 4\,d_1 d_3$, which remarkable relation between the diameters of the two circles and their common tangent holds true, in general, for the two circles that can be described touching any three circles that touch each other. I proposed this remarkable theorem in an old number of "*The Educational Times;*" but no geometrical demonstration of the general case has yet appeared. The foregoing particular case is that in which one of the three circles that touch each other becomes infinite.

21 Eden Quay, Dublin, Ireland, February 26, 1859.

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Note. — We commend this very interesting article to the attention of our readers. Its fundamental propositions depend so directly upon the Elements of Geometry as given in all our text-books, that it will be read with ease by all. Those who wish to pursue this subject, including the more general problem of contact, will find it treated by the following authors: Pappus, Vieta, Descartes, Newton, Euler and Fuss (*Memoirs de l'Acad. de Pétersbourg*, 1788); Monge, (*Correspondance sur l'École*

*Polyt.*, tomes I. et II.); GAULTIER DE TOURS (*Journal de l'Ecole Polyt.* 1813); GERGONNE (*Memoires de l'Acad. de Turin*, 1814, *Annales de Math.*, tomes IV., VII., XI.); DURRANDE and PONCELET (*Annales de Math.*, tome XI.), LESLIE, PUISSANT, STEINER (CRELLE'S *Journal*, tome I., *Annales de Math.*, tome XVII); *Library of Useful Knowledge*, Vol. on Geometry, ALVORD (Smithsonian Contributions). This list, which is by no means complete, shows the amount of research which has been devoted to this subject by many of the ablest geometers.

The term, *Centre of Similitude*, was introduced by MONGE, and that of *Radical Axis* by GAULTIER DE TOURS.

In regard to the theorem 5th, page 276, it may be stated that a paper on this subject was read March 9th, 1858, before the American Academy of Arts and Sciences, by Mr. J. B. HENCK. In this paper, which is mentioned in the Proceedings of the Academy, and which will probably appear in its Memoirs, the above theorem, with other properties of tangent circles, is demonstrated and extended to the case in which one of the fixed circles is tangent to the other externally, as well as to the case in which it becomes a straight line.

It may be remarked also, that PAPPUS, in Book IV., Theorem XV., of his *Mathematicæ Collectiones*, demonstrates that (COLLINS' Fig. 8, page 276), $OF + 2\,Of : 2\,Of :: O'G : 2\,O'g$, which, if $OF = n \times 2\,Of$, becomes $n \times 2\,Of + 2\,Of : 2\,Of :: n+1 : 1 :: O'G : 2\,O'g$. Of this interesting theorem PAPPUS gave two special cases worthy of note. (1) If $n = 0, 1, 2, 3$, &c., we obtain the corollary on page 277, the case in which the first of the series of inscribed tangent circles is described on $BC$ as a diameter. This case is PAPPUS' Theorem XVI., and was referred to by him as the *Arbelos*. (2) If $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, &c., we have the case in which the first circle is tangent to $BC$, which is PAPPUS' Theorem XVIII.

It is a little singular, that LESLIE, devoted as he was to the ancient geometry, while demonstrating the first of these cases, should have entirely omitted the second, which is equally remarkable.

In GILL'S *Mathematical Miscellany*, a problem is given which requires the sum of the areas of all the inscribed tangent circles. Solutions are given by the Editor, and Dr. THEODORE STRONG, both of which incidentally develop many interesting properties.

## NOTE ON THE COURSING JOINT CURVE OF AN OBLIQUE ARCH IN THE FRENCH SYSTEM.

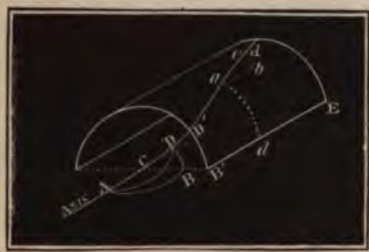
BY DEVOLSON WOOD, C. E.,
Assistant Professor of Engineering in the University of Michigan.

THE equation of the curve, as given by me in the March number of the MONTHLY, should contain $\frac{1}{\sin\theta}$ as a factor, instead of $\cot\theta$.

The differential triangle $a\,b\,c$ is right-angled at $c$, instead of $b$, and hence we should have

$$\frac{b\,c}{a\,c} = \frac{r\,d\,\varphi}{d\,s} = \tan i.$$

But to find the equation in terms of $x$ instead of $s$, let a right section be made through $b$, and the angle which this section makes with the section parallel to the face is $\theta$. Hence $b\,d$, which is the perpendicular of the right-angled triangle $a\,b\,d$, equals $\frac{b\,c}{\cos\theta} = \frac{r\,d\,\varphi}{\cos\theta}$.




$$\therefore \qquad \frac{b\,d}{a\,b} = \frac{r\,d\,\varphi}{\cos\theta\,d\,x} = \tan i = \sin\varphi\,\tan\theta,$$

which gives $$d\,x = \frac{r\,d\,\varphi}{\sin\theta\,\sin\varphi};$$

or by integration $$x = \frac{r}{\sin\theta}(\log\tan\tfrac{1}{2}\varphi - \log\tan\tfrac{1}{2}\varphi_1).$$

Now, for $\varphi_1 =$ any finite value, $x$ will have some finite value for $\varphi = \frac{\pi}{2}$; hence every coursing joint will pass through the crown, if sufficiently prolonged. Take the origin of coördinates such that the initial value of $\varphi_1 = \frac{\pi}{2}$, and the equation becomes

$$x = \frac{r}{\sin\theta}\log\tan\tfrac{1}{2}\varphi.$$

If $\varphi_1$ be the superior limit, the equation will become

$$x = \frac{r}{\sin\theta}(\log\tan\tfrac{1}{2}\varphi_1 - \log\tan\tfrac{1}{2}\varphi).$$

Now if $\varphi_1 = \frac{\pi}{2}$, we have

$$x = -\frac{r}{\sin\theta}\log\tan\tfrac{1}{2}\varphi = \frac{r}{\sin\theta}\log\cot\tfrac{1}{2}\varphi,$$

which is the same as that given on page 210; and only lacks the constant of integration.

## EQUATION OF THE COURSING JOINT CURVE.

BY WILLIAM G. PECK,
Adjunct Professor of Mathematics in Columbia College, New York.

THE equation of the developed "coursing joint curve" may be found as follows: Assuming the figure and general notation of page 209, we have for the equation of a developed ring joint,

$$y = a \mp r \operatorname{versin} \varphi \sin \theta, \tag{1}$$

in which $a$ is arbitrary, corresponding to any ring joint. The upper sign corresponds to the case represented in the figure in which the arch skews to the right, and the lower one to the case in which it skews to the left. There will consequently be two equations. The equation of the developed coursing joint will be of the form

$$f(x, \varphi) = 0. \tag{2}$$

Differentiating (1) we find

$$\frac{dy}{d\varphi} = \mp r \sin \theta \sin \varphi. \tag{3}$$

The condition that the curves (1) and (2) are to be normal to each other requires that

$$r^2 + \frac{dy}{d\varphi} \times \frac{dx}{d\varphi} = 0,$$

or substituting for $\frac{dy}{d\varphi}$ its value taken from (3), we have

$$r^2 \mp r \sin \theta \sin \varphi \frac{dx}{d\varphi} = 0,$$

whence
$$dx = \pm \frac{r}{\sin \theta} \cdot \frac{d\varphi}{\sin \varphi}.$$

Taking the upper sign and integrating, we have

$$x = \frac{r}{\sin \theta} \log \tan \tfrac{1}{2} \varphi + c.$$

Taking the lower sign and integrating, we have

$$x = \frac{r}{\sin \theta} \log \cot \tfrac{1}{2} \varphi + c.$$

The latter equation is the same as that on page 210.

## REMARKS UPON CAYLEY'S (SUPPOSED) NEW THEOREM OF SPHERICAL TRIGONOMETRY.

BY W. CHAUVENET,
Professor of Mathematics in the United States Naval Academy, Annapolis, Md.

THE February number of the *Philosophical Magazine* contains a note by Mr CAYLEY, in which he gives as *new*, the following equation or theorem of Spherical Trigonometry:

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a,$$

and the March number of the same magazine contains a demonstration of the theorem by the Astronomer Royal. Mr. CAYLEY gives an analytical demonstration of it; Mr. AIRY, a geometrical or semi-geometrical demonstration.

It is remarkable that a standard work like CAGNOLI'S *Trigonometrie*, published in French, in 1808, should be so little known in England that this theorem, which is the especial property of CAGNOLI, should, after half a century, be treated as new by mathematicians like CAYLEY and AIRY. Not only does CAGNOLI give the theorem (in Art. 1139 of his Trigonometrie) together with its demonstration; but he also makes an application of it (in Art. 1135 of the same work) to the solution of the problem of finding the aberration in declination of the fixed stars.

But the theorem might have been found also in DELAMBRE'S *Astronomie* (Vol. I., Chap. X., Art. 77), where it is ascribed to CAGNOLI, but differently proved. It is a singular coincidence, that CAYLEY'S analytical demonstration is essentially the same as CAGNOLI'S; and AIRY'S geometrical demonstration the same as DELAMBRE'S.

There is one property of this equation which deserves notice here, although it is sufficiently obvious to have attracted the attention of others. It is, that if one member of it is applied to the polar triangle, it gives the other member; that is, *either member is a function*

*which has the same value in the given spherical triangle and in its polar triangle.* Thus, if we denote the sides and angles of a spherical triangle by $a, b, c, A, B, C$, and those of its polar triangle by $a', b', c', A', B', C'$, we have

$$\sin b \sin c + \cos b \cos c \cos A = \sin b' \sin c' + \cos b' \cos c' \cos A'.$$

There are several other such functions in spherics. Thus we have

$$\sin b \sin c \sin^2 A = \sin b' \sin c' \sin^2 A'.$$

If $n$ is a function such that

$$n^2 = \sin s \sin (s - a) \sin (s - b) \sin (s - c),$$

in which $s = \frac{1}{2}(a + b + c)$, and if $n'$ is the same function of the sides of the polar triangle; then

$$n \sin A = n' \sin A'.$$

We have also

$$\cos \frac{a}{2} \sin \frac{B+C}{2} + \sin \frac{a}{2} \cos \frac{B-C}{2} = \cos \frac{a'}{2} \sin \frac{B'+C'}{2} + \sin \frac{a'}{2} \cos \frac{B'-C'}{2},$$

and others of the same kind, which, however, are too complicated to be useful, and are not worth repeating.

# A SECOND BOOK IN GEOMETRY.

## REASONING UPON FACTS.

BY THOMAS HILL.

[Continued from page 256.]

## CHAPTER IV.

### ANALYSIS AND SYNTHESIS.

41. WHAT I have called demonstration or deduction, but which is better called synthesis, because it is a putting together, one by one, of the parts of a complex truth, is the only mode of proof that you will usually find in works on geometry. And if such works are carefully read they are always intelligible to a child of good geometrical reasoning powers.

42. But the study of such works does not always teach a child to reason for himself. The pupil says, "Yes, I understand all this, and yet I could not have done it without aid; I do not see how the writer knew where to begin; how he knew that by starting from these particular truths, and going on that particular path, he could reach that proposition." A pupil who had never studied geometry could not, for instance, tell why in articles 34–36 we should begin with showing that vertical angles are equal. He would not see any connection between that truth, and the desired proof; and would not know that this synthesis had been preceded, in the mind of the writer, by a rapid analysis, such as that of Arts. 26–31.

43. It is as though a mountain guide, wishing to make for a child a path up to a mountain peak, should lead him along the highway, until the peak was hidden, and then begin boldly to clear a road, through the brush-wood and trees, until he reached the top. The child might say, "How did you dare begin at once to cut down the bushes and clear the path? How did you know that the road you were making would not lead you to the edge or to the foot of some precipice, or that it would not take you to a different peak from that which you wished to climb?" And if the child received no answer to his questions, — if he was not told that the guide had already climbed to the summit and again descended, he would have learned little to help him in laying out paths for himself.

44. In like manner, although the descent from difficult propositions to more simple is more tedious than the ascent, it will be more useful to a learner, because it will show him the manner in which, by a mental process, we discover the points from which we are to start in our ascent. This is to say, if we follow a good analysis, we shall learn how to perform synthesis for ourselves; but if we were simply to follow a writer's synthesis, we should not learn how to analyze, which must nevertheless always go before synthesis.

45. Among the first requisites in reasoning is a clear understanding of the object in view; that is, of the point to be proved; and next, a clear perception of each particular part of the demonstration, and of the connection of each part with the adjacent parts.

Thus in laying out a path up a mountain, it is necessary to know exactly from what point you wish to start, and to what point you wish to go. It is also necessary to examine carefully each point of the road, for a single impassable place would destroy the value of the whole road.

46. Each step of the proof must be a simple step, and clearly true; that is, it must be so simple and self-evident as to be beyond all doubt.

47. The analysis must end, or the synthesis begin, with truths that are self-evident, or else that have been already proved. Your mountain path must begin on level or at least on accessible ground.

48. Care must be taken not to introduce any thing as true which has not been proved. This would be like starting your mountain road in two places at once. You might afterwards find impassable barriers between the two parts of your road, and perhaps find that one of them could not be made to the top of the mountain, nor the other to its base. For example, in Art. 36, I drew a straight line through A, parallel to B C. This was very well — for no one can possibly doubt such a line might be drawn. But if, instead of that I had said, let us draw a straight line through A in such a manner as to make the angles on the two sides of A equal to the angles B and C, I should have done what I had no right to do. For that would be taking for granted a thing which I must prove: namely, that a straight line can be thus drawn. It would be starting half way up my mountain, and taking for granted that the lower part of the path could be built afterwards.

49. Whether we reason by synthesis or analysis, we must therefore reason very carefully,

in order to connect the proposition which we wish to prove by a stairway of self-evident steps with a self-evident foundation.

50. By a self-evident truth, I mean a truth which cannot be made any plainer, and which is already perfectly plain to an intelligent person who looks steadily at it. For instance, that two straight lines can cross each other only once; that any curve can be cut by a straight line in at least two places; that either side of a triangle is shorter than the sum of the other two; that if three strings, and no more than three, come from one point, one of them must have an end at that point; these are self-evident truths.

51. By a self-evident step in reasoning, I mean the statement of the relation of one truth to another, or of the dependence of one truth upon another, when that dependence or that relation is itself a self-evident truth. Self-evident steps in reasoning are simply the statement of self-evident truths of connection. For instance, when we have explained the meaning of "a straight line" by calling it a line that has in every part the same direction, and have explained the meaning of an angle by the difference of two directions in one plane, then it follows that the angle which two straight lines make with each other is the same in one part of the lines as in any other; and that the two different angles apparently made by two straight lines cannot really be made, unless one of the lines goes in two opposite directions at the same time. No reasoning can make the connection between these definitions and the equality of vertical angles any more plain. It is a self-evident connection.

C

52. Or, suppose that we say that you cannot make one rope go from a centre post to the four corners of a square, and also around the square, and have but a single rope from post to post. We should prove it in this way. Let there be a rope around the square, and going also from each post to the centre. This of course can be imagined. It is a definite and allowable conception. But we will also prove that this rope must be in two pieces. For each of the four corners will have three lines coming from it, one towards each adjacent corner, and one towards the centre. Thus it follows by self-evident connection, from the conception of the rope going around the square and to each corner, that there will be four points, from each of which three lines come. But it is a self-evident truth that at each of these points there must be one end of the rope. Hence, by self-evident connection, there will be four ends of rope about the square. Hence, by self-evident connection with the self-evident truth that one piece of rope can have but two, and must have two, ends, it follows that there must be two pieces of rope, and cannot be only one. Now the whole of this proof is simply the statement of self-evident connections between the proposition that one rope cannot go around a square and also from each corner to the centre without doubling, and the self-evident truths that a piece of rope must have two, and cannot have more than two, ends; and that when only three lines of rope come from one point, one of them must end at that point. The proof is simple; and yet intelligent men have spent hours in experimenting with a string and five posts thus arranged; or with a pencil and five dots representing posts.

53. Many self-evident truths are general, and self-evident steps are generally the recognition of general relations; and therefore most writers on reasoning, say that reasoning consists simply in showing that a particular case comes under a general class. But in the mathematics, there are many self-evident truths which it is difficult to state in a general form; and I therefore think that the explanation which I have given of the process of reasoning, will be of more use to you in your geometrical studies.

# Mathematical Monthly Notices.

*Elements of Mechanics; for the Use of Colleges, Academies, and High Schools.* By WILLIAM G. PECK, M. A., Adjunct Professor of Mathematics, Columbia College. New York: A. S. Barnes & Burr, 51 and 53 John Street. 1859.

THE standard the author of this work had in mind, though not the method of demonstration employed in its preparation, is sufficiently obvious from the title-page, even if his preface had not contained the formal statement that it is intended to occupy middle ground between works on Natural Philosophy of mere description, without any attempt at rigorous demonstration, and those demanding considerable knowledge of the higher mathematics.

The question which at once suggested itself to us was, ought the author to have used the principles of the calculus? If it was intended that the work should be studied after the Elements of the calculus, there could be no question about it. But if, on the other hand, it is to be put into the hands of those who have not studied the calculus, however thoroughly they may have mastered the more elementary branches, then it is as certain that the calculus should not be used. The author has, therefore, in using the calculus in the chapters "On the Centre of Gravity," "On Motion," &c., limited the use of his work by so doing. We should regret this the less, if we felt less pleased with it. When the elements of the higher mathematics are as generally studied as they should be, in our institutions of learning, then Prof. PECK's work will be well adapted for the use of Colleges, Academies, and High Schools. Now, as a whole, it is only adapted for use in those institutions in which the study of the calculus is a part of the required course, and is taught to all; but this is less a criticism of the book than of the present mathematical standard in our system of education.

In deciding what subjects to introduce, in what order to arrange them, and how general a discussion to give them, we think the author has shown good judgment. If, in the few cases investigated by the calculus, an analysis had been used which the student of elementary geometry or algebra could comprehend, and the demonstrations, at present in the text, were in the margin, thus adapting the work to both classes of pupils, we hardly know in what other respect it could be materially improved; unless it be, as it seems to us, to add under each head a few more difficult examples to the judicious selection already given. The calculus is an elective study in some of our best colleges, and when such is the case but few indeed elect it. In academies and high schools, the study is always elective, so far as we know, and never studied at all in such institutions, unless the teacher happens to be a good mathematician himself, and is ambitious to teach a whole course of pure mathematics; and even in such a very favorable case, only a very small class at best can usually be found prepared to take it. To be sure, only its very first and simplest principles are used in the book before us; but this does not in the least remove the difficulty, since the notation even of the calculus is peculiar, and its fundamental conceptions equally new to one whose mathematical knowledge is confined to the simpler elements of mathematics. We know it has been, and is still, a question with good teachers of mathematics, whether it is better for the student, when the subject demands the calculus, to take a work which avoids it by resort to methods, which, however general they may be in their character and uniform in their application, no one conversant with the

calculus would ever think of using as a means of investigation; or whether he will not in the end be by far the gainer in all respects by first mastering the elements of the calculus, and afterwards using it in all investigations demanding the infinitesimal analysis. WEISBACH'S *Treatise on Mechanics and Engineering*, in which the calculus is not used, but in which ability of a high order, with large experience, is taxed to furnish the best possible substitute, is a good case in point. We are free to admit, that we should much prefer, and we think it would be altogether best for the student, to teach him the calculus first, and then a WEISBACH written in its language and symbols; and we hope yet to see this branch much more generally studied in our schools, and introduced much earlier in the course. But this is hardly to be expected so long as the standard for admission to college is so low, and especially as compared with the ancient languages. This inequality in the preparatory course is even more unfortunate for the mathematics after than before admission to college.

*A Tract on the Possible and Impossible Cases of Quadratic Duplicate Equalities in the Diophantine Analysis*: To which is added a short but comprehensive Appendix, in which most of the useful and important Propositions in the Theory of Numbers are very concisely demonstrated. By MATTHEW COLLINS, B. A., Senior Moderator in Mathematics and Physics, and Bishop Law's Mathematical Prizeman, Trinity College, Dublin.

The following synopsis of the Contents of this Tract will give the best idea of its character; and to those of our readers interested in this department of analysis we most heartily commend it.

The possible and impossible cases of the following sets of two simultaneous equations are treated, namely:

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|---|---|
| CHAP. I. | $x^2 + a y^2$ and $x^2 - a y^2 =$ squares. |
| CHAP. II. | $x^2 + y^2$ and $x^2 + a y^2 =$ squares. |
| CHAP. III. | $x^2 + y^2$ and $x^2 - a y^2 =$ squares. |

By one uniform method it is proved that the first set is impossible for all integer values of $a$ less than 20, except 5, 6, 7, 13, 14, or 15; the second set is impossible for any integer values of $a$ between 1 and 20, except 7, 10, 11, or 17; and the third set for any integer values of $a$ between 1 and 18, except 7 or 11. General formulas are given, much shorter than FERMAT'S method, for finding any number of solutions in integers prime to each other.

Chapter II. also contains a scholium, giving very extensive and important additions to the few cases in which the equation

$$a x^4 + b x^2 y^2 + c y^4 = \text{a square},$$

was heretofore known and proved to be impossible by FERMAT and EULER.

The Tract closes with a list of over three hundred and fifty subscribers, including the names of HAMILTON, GRAVES, JELLETT, LARDNER, WOOLHOUSE, and others, well known in this country; and those wishing to procure it will consult the third page of cover.

# Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the February number of the MONTHLY: —

GEORGE B. HICKS, Student, Cleveland, Ohio, answered all the questions.

ARTHUR H. WRIGHT, Senior Class, Yale College, answered all the questions. (H. A. NEWTON, Prof.)

JOHN Y. BEDINGFIELD, Student, Bowdon Collegiate Institution, Bowdon, Carroll Co., Ga., answered all the questions. (JOHN M. RICHARDSON, Prof. Math.)

A Student in Union Square Academy, Baltimore, Md., answered all the questions but III. (JOHN McNEVIN, Prof.)

ASHER B. EVANS, Junior Class, Madison University, Hamilton, New York, answered all the questions. (L. M. OSBORN, Prof.)

WILLIAM E. MERRILL, Cadet, First Class, U. S. Military Academy, West Point, N. Y., answered all the questions. (A. E. CHURCH, Prof.)

WALLER HOLLADAY, Student of Mathematics in the University of Virginia, answered all the questions. (A. T. BLEDSOE, Prof.)

F. RICHARDSON, Junior Class, Haverford College, Delaware Co., Pa., answered all but questions III and V. (M. C. STEVENS, Prof.)

JOHN R. MEIGS, Junior Class, Columbian College, Washington, D. C., answered all but questions III and V. (EDWARD T. FRISTOE, Prof.)

GEORGE A. OSBORNE, Jr., Student in the Lawrence Scientific School, answered all the questions. (H. L. EUSTIS, Prof.)

DAVID TROWBRIDGE, Student, Perry City, Schuyler Co., N. Y., answered all the questions but I.

ISAAC H. HALL, Senior Class, Hamilton College, Clinton, N. Y., answered all the questions. (OREN ROOT, Prof.)

GEORGE W. JONES, Jr., Senior Class, Yale College, answered all the questions. (H. A. NEWTON, Prof.)

It gives us pleasure to add the following names to our list of coöperators and contributors: S. E. BENJAMIN, Esq., Patten, Me.; JAMES CLARK, Esq., Wayne, Me.; THOMAS P. STOWELL, Esq., Hornellsville, Steuben Co., N. Y.; CHARLES D. LAWRENCE, Professor of Mathematics, Bethel College, Russellville, Ky.; BENJAMIN ALVORD, Major U. S. Army, Fort Vancouver, Washington Territory; Dr. WILLIAM J. WALLER, President of Shelby College, Shelbyville, Ky.; JOHN BORDEN, Esq., Chicago, Ill.; Rev. ANTHONY VALLAZ, Phil. Dr., Late Ordinary Professor of Mathematics in the Royal University of Pesth, New Orleans, La.; JOHN A. NICHOLS, Professor of Mixed Mathematics in the Free Academy, New York City; ADOLPH WERNER, Esq., N. Y. Free Academy. . . . . . We have more material on hand accepted for publication, than is contained in the eight numbers of the MONTHLY already issued. This will explain why contributions are so often delayed beyond the month for which their authors intended them. This is a matter entirely out of our control, and must remain so, while we are obliged to confine the MONTHLY to its present number of pages per month.

THE

# MATHEMATICAL MONTHLY.

Vol. I....JUNE, 1859....No. IX.

## PRIZE PROBLEMS FOR STUDENTS.

I.

In a right-angled triangle, having given the difference between the base and perpendicular, and also the difference between the hypothenuse and base; to construct the triangle geometrically.

II.

In a right-angled triangle, having given the sum of the base and perpendicular, also the sum of the hypothenuse and base; to construct the triangle geometrically.

III.

The four tangents, which are common to two circles which do not intersect, and are terminated at their points of respective contact, have their middle points on the radical axis of the two circles.

IV.

The external centres of similitude of three circles, taken successively two and two, all lie in one straight line; and each of them is situated in a right line with two of the internal centres of similitude.

V.

Let two circles be touched respectively by a single straight line $AA'$ in $A$ and $A'$, and by a single circle in $BB'C$ in $B$ and $B'$; if

the straight line and the circle touch in the same manner the two circles, the point $C$ of the meeting of $AB$ and $A'B'$ will lie on the circumference of the circle $BB'C$ and on the radical axis of the two other circles.

The solution of these problems must be received by the first of August, 1859.

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### REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. V., Vol. I.

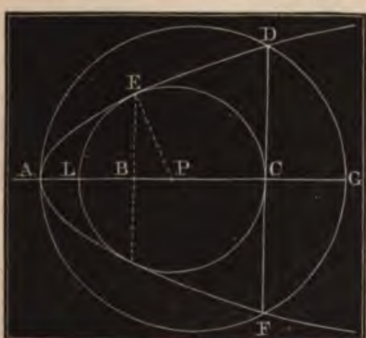
The first Prize is awarded to Arthur W. Wright, of the Senior Class in Yale College, New Haven, Ct.

The second Prize is awarded to George A. Osborne, Jr., of the Lawrence Scientific School, Cambridge, Mass.

#### Prize Solution of Problem I.

"The abscissa and double ordinate of a segment of a common parabola are $a$ and $b$, and the diameters of its circumscribed and inscribed circles $D$ and $d$; to prove that $D+d=a+b$."

Let $A$ be the origin of coördinates, and $AG$ the axis of $x$,




$AC=a$, and $DF=b$. The equation of the parabola is $y^2=2px$, and that of a circle having its centre on the axis of $x$ is $(x-x_0)^2+y^2=r^2$. If this circle is tangent to $DF$, then we have $x_0=AC-PC=a-r$, and its equation becomes $(x-a+r)^2+y^2=r^2$. Combining this with the equation of the parabola, we get $(x-a+r)^2+2px=r^2$; therefore, $x=a-r-p\pm\sqrt{(r+p)^2-2pa}$, which gives the abscissas of the points of intersection of the two curves. But when the circle becomes tangent to the parabola, these values of $x$ must be equal,

and hence $\sqrt{(r+p)^2-2pa}=0$. $\therefore r=-p+\sqrt{2pa}=-p+\frac{1}{2}b$, since $2pa=\frac{1}{4}b^2$. Therefore $2r=d=b-2p$.

The equation of the circle circumscribing the segment is $y^2=2Rx-x^2$, which, combined with that of the parabola, gives $2px=2Rx-x^2$; and for $x=a$, $2R=D=a+2p$. Therefore $D+d=a+b$.

This solution is by Arthur W. Wright.

Note. For $E$, the point of tangency, $x=a-r-p=AC-PC-BP$, and therefore $p=BP$; that is, the subnormal is constant and equal to the semi-parameter. Upon this property most of the solutions were based. Thus, in the right-angled triangle $EBP$, we have $r^2=y^2+p^2$; but $y^2=2px=2p(a-r-p)$; and therefore $r^2=2p(a-r-p)+p^2$. Hence $r=-p\pm\sqrt{2pa}=-p\pm\frac{1}{2}b$. And since $DC^2=AC\times CG$, or $\frac{1}{4}b^2=2pa=(2R-a)a$, therefore $2R=a+2p$.

It will be observed that there are two values of $r$; namely, $-p+\frac{1}{2}b$, and $-p-\frac{1}{2}b$, the first of which corresponds to the circle $LEC$. But a circle tangent to the parabola may be drawn tangent to $DF$ on the side opposite to $A$; and if $x_0=a+r$, then $r=p\pm\frac{1}{2}b$; and as these values of $r$ are the same as those above with changed signs, it follows that the same solution gives the numerical values of the radii of both the tangent circles. The second tangent circle was noticed by George A. Osborne, Jr.

When the inscribed circle is tangent to the parabola at $A$, the abscissa of the point of tangency is zero; that is $x=a-r-p=0$, or $a-\frac{1}{2}a-p=0$, or $a=2p$; which is the least value of $a$ for which the problem holds. This limitation was noticed by George W. Jones, Jr., and Arthur W. Wright.

### Prize Solution of Problem II.

"A great circle of the sphere passes through two given points; find the rectangular coördinates of its pole."

The equation of the sphere, referred to its centre as origin, is $x^2+y^2+z^2=r^2$ (1). Let $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ be the coördinates of the two given points, and $(x, y, z)$ those of the required pole. Since the axis of the great circle, containing the given points, is perpendicular to its plane, it will be perpendicular to each of the radii containing these points; and also since $\left(\frac{x_1}{r}, \frac{y_1}{r}, \frac{z_1}{r}\right)$, $\left(\frac{x_2}{r}, \frac{y_2}{r}, \frac{z_2}{r}\right)$ and $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ are the cosines of the angles which the two radii and the axis of the circle make with the coördinate axes; we have

$$(2) \qquad \frac{x_1}{r}\cdot\frac{x}{r}+\frac{y_1}{r}\cdot\frac{y}{r}+\frac{z_1}{r}\cdot\frac{z}{r}=0, \text{ or } x x_1+y y_1+z z_1=0;$$

$$(3) \qquad \frac{x_2}{r}\cdot\frac{x}{r}+\frac{y_2}{r}\cdot\frac{y}{r}+\frac{z_2}{r}\cdot\frac{z}{r}=0, \text{ or, } x x_2+y y_2+z z_2=0.$$

From (2) and (3) we obtain

$$(4) \qquad y=\frac{x_1 z_2-x_2 z_1}{z_2 y_2-z_2 y_1}\,x, \text{ and } (5)\ z=\frac{y_1 x_2-y_2 x_1}{z_1 y_2-z_2 y_1}\,x,$$

which are the equations to the axis of the great circle. Combining (4) and (5) with (1), we have

$$x^2+\left(\frac{x_1 z_2-x_2 z_1}{z_1 y_2-z_2 y_1}\right)^2 x^2+\left(\frac{y_1 x_2-y_2 x_1}{z_1 y_2-z_2 y_1}\right)^2 x^2=r^2;$$

or $[(z_1 y_2-z_2 y_1)^2+(x_1 z_2-x_2 z_1)^2+(y_1 x_2-y_2 x_1)^2]\,x^2=(z_1 y_2-z_2 y_1)^2 r^2$.

The first member of this equation can be reduced by observing that

$$(z_1 y_2-z_2 y_1)^2+(x_1 z_2-x_2 z_1)^2+(y_1 x_2-y_2 x_1)^2=(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)$$
$$-(x_1 x_2+y_1 y_2+z_1 z_2)^2=r^4-(x_1 x_2+y_1 y_2+z_1 z_2)^2=r^4-r^4\cos^2\varphi=r^4\sin^2\varphi,$$

in which $\varphi$ denotes the angle made by the two radii containing the given points.

Hence we have $x^2 r^4 \sin^2\varphi=(z_1 y_2-z_2 y_1)^2 r^2$; or $x=\pm\frac{z_1 y_2-z_2 y_1}{r\sin\varphi}$.

Similarly it may be shown that $y=\pm\frac{x_1 z_2-x_2 z_1}{r\sin\varphi}$, and $z=\pm\frac{y_1 x_2-y_2 x_1}{r\sin\varphi}$.

This solution is by George A. Osborne, Jr.

### Prize Solution of Problem III.

"If the two sides of a movable right angle are always tangents to a given ellipse, its summit will describe a circle concentric with the ellipse, the radius of which is equal to the chord joining the extremities of the major and minor axes."

This is simply to find the locus of the intersection of pairs of tangents which are at right angles to each other. Let the equation of the given ellipse be $A^2 y^2 + B^2 x^2 = A^2 B^2$ (1), and $\alpha, \beta$ the coördinates of any point in the locus. The equation of a straight line passing through $\alpha, \beta$ is $y - \beta = m(x - \alpha)$ (2).

Let $y$ be eliminated between (1) and (2), and the result arranged according to powers of $x$; then

$$(A^2 m^2 + B^2) x^2 + 2 A^2 m (\beta - m \alpha) x + A^2 (\beta^2 + \alpha^2 m^2 - 2 \beta \alpha m - B^2) = 0.$$

In order that (2) may be a tangent to (1), the values of $x$ from this quadratic must be *equal*. Whence by the theory of equations

$$[2 A^2 m (\beta - m \alpha)]^2 = 4 A^2 (A^2 m^2 + B^2) (\beta^2 + \alpha^2 m^2 - 2 \beta \alpha m - B^2).$$

Developing this expression, reducing and arranging according to powers of $m$,

$$m^2 + \frac{2 \alpha \beta}{\alpha^2 - A^2} m + \frac{\beta^2 - B^2}{\alpha^2 - A^2} = 0.$$

The two values of $m$ from this equation can only belong to the two tangents which can be drawn to the ellipse through $\alpha, \beta$. Calling one value $m$, and the other $m'$, the theory of equations gives the following relation:

$$m m' = \frac{\beta^2 - B^2}{A^2 - \alpha^2};$$

and, since the tangents are at right angles to each other, $m m' = -1 = \frac{\beta^2 - B^2}{\alpha^2 - A^2}$. Whence $\alpha^2 + \beta^2 = A^2 + B^2$, and therefore the locus is a circle.

This solution is by George B. Hicks.

### Prize Solution of Problem IV.

"If a circle be described through the foci of an ellipse and any point in the con-

jugate axis produced; to prove that the right line joining that point and one of the points where the circle cuts the ellipse will be a tangent to the ellipse."

Let the circle $F' E F$, passing through any given point as $E$ in


the conjugate diameter of the ellipse $A' B A$, and also through its foci, meet the curve in the point $P$. Join $P E'$, and from $P$ draw $P F$ and $P F'$ to the foci $F$ and $F'$, and from $E$ through $P$ draw $E P$. Since $E E'$ is perpendicular to $A A'$, and $CF = CF'$, the arc $FE'$ is equal to the arc $F' E'$, and therefore the angles $FPE'$ and $E' P F'$ are equal. But $E P E'$ is a right angle, since the arc $E P E'$ is a semicircle. Hence the angles $TPF$ and $EPF'$ are equal, and therefore $EPT$ is tangent to the ellipse.

This solution is by Arthur W. Wright; and most of the others are essentially the same.

### Prize Solution of Problem V.

"If $D$ represent any diameter of an ellipse, and $P$ the parameter of $D$, to find when $D + P$ is the least, and when the greatest, possible."

Since the sum of the squares of any two conjugate diameters equals the sum of the squares of the axes, $D'^2 + D^2 = 4\,(A^2 + B^2)$. But $P = \frac{D'^2}{D}$, and $D + P = D + \frac{D'^2}{D} = \frac{D^2 + D'^2}{D} = \frac{4\,(A^2 + B^2)}{D}$. Hence, since $A^2 + B^2$ is constant, $D + P$ varies inversely as $D$, and is greatest when $D$ is least, or $2\,B$, and least when $D$ is greatest, or $2\,A$.

This is substantially the solution given by nearly all the competitors.

Joseph Winlock.
Chauncey Wright.
Truman Henry Safford.

## CONSTRUCTION OF A PROBLEM.

By J. E. Hilgard, United States Coast Survey, Washington, D. C.

Problem. *Given, an elliptical right cone, to construct the angle made with its axis by a plane which intersects it in a circle.*

The solution consists in describing a sphere from any point of the axis of the cone as centre, and tangent to the longest sides of the cone. This sphere will intersect the surface of the cone in two circles, the planes of which are determined by the intersections of the sphere with the shortest sides of the cone.

Let $SAB$ be a section through the major axis of the base, and $SDE$ a section through the minor axis. From any point $C$ of the axis inscribe in the angle $ASB$ the circle $LHG$; then the lines $FG, HK$, joining the intersections of the circumference with the opposite sides of the section $SED$, will give the required angle with the axis.


This elegant solution is due to Prof. Engel. Its demonstration is left to the student.

The method is of general application to surfaces of the second order having circular sections, such as the oblique elliptical cone, the ellipsoid of three axes, the elliptical and hyperbolical hyperboloids, &c.

## QUESTION, BY MATTHEW COLLINS, B. A., DUBLIN, IRELAND.

If we multiply a circulating decimal (pure or mixed) having $m$ figures in its period by another such circulate having $n$ figures in its period, prove that the product will be a circulate having $9\,m\,n$ figures in its period if $m$ be prime to $n$. But if $n$ be equal to $m$

or to a multiple of $m$, prove that the period in the product will then have $n(10^m - 1)$ figures in it; if $m = 4$ and $n = 6$, prove that the period in the product will consist of $99 \times 6 \times 2$, or 1188 figures; also extend and generalize this latter part of this new and curious theorem.

## PROPOSITIONS RELATING TO THE RIGHT-ANGLED TRIANGLE.

BY M. L. COMSTOCK,
Assistant Professor of Mathematics in Knox College, Galesburg, Illinois.

I SEND you a few additional propositions relating to right-angled triangles, which may be useful to those who require original demonstrations of their pupils.

1. In any right-angled triangle, if the circumference of a circle be drawn through the extremities of the hypothenuse and the centre of the inscribed circle; the diameter of the circle so drawn will be the diagonal of the square described on the hypothenuse, and its centre will be the circumference of the circle circumscribing the triangle.

2. If a circumference of a circle be drawn through both extremities of either leg and the centre of the inscribed circle, its centre will be on the circumference of the circumscribing circle.

3. If the lines which bisect the acute angles of a right-angled triangle be drawn, and produced to meet the circumference of the circumscribing circle, the rectangle of the parts intercepted between the angles and the opposite sides of the triangle is equal to four times the rectangle of the lines intercepted between the centre of the inscribed, and circumference of the circumscribing, circles.

4. Also the first-mentioned rectangle is equal to twice the rectangle of the lines intercepted between the acute angles and centre of the inscribed circle.

# NOTE ON THE CYCLOID.

BY LEWIS R. GIBBES,
Professor of Mathematics in the College of Charleston, South Carolina.

In many elementary treatises on Mechanics, the student is informed that the cycloid is the curve of quickest descent under the restrictions usually given, but the demonstration of this property of the cycloid is supposed to be inaccessible to him, without a knowledge of the calculus of variations. The following geometrical demonstration will be valued by the student whose course of study does not extend so far, and perhaps may not be contemned by those who are familiar with all the resources of the calculus. We do not believe it possible to present the demonstration in a simpler form. We will first premise the three following propositions easily demonstrable.

Prop. 1. If two right-angled triangles have the same altitude, the product of the sum and difference of their hypothenuses is equal to the product of the sum and difference of their bases. Easily deducible from the well-known theorem of the relation of the sides of a right-angled triangle.

Prop. 2. The velocity at any point of descent in a curve is equal to the velocity due to the vertical fall or height; which velocity is proportional to the square root of that height. Given in the treatises.

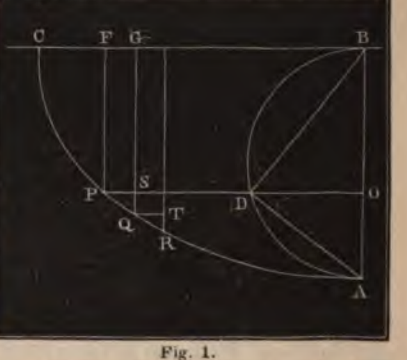


Fig. 1.

Prop. 3. At any point in the cycloid,

$$\frac{\text{the increment of the ordinate,}}{\text{the increment of the curve} \times \sqrt{\text{axis} - \text{abscissa}},}$$

is a constant quantity, or, what is the same, such quantities for any two points in the curve are equal; that is, in Fig. 1, $\frac{PS}{PQ\sqrt{PF}} = \frac{QT}{QR\sqrt{QG}}$ the origin of coördinates being at the vertex $A$.

By the properties of the cycloid, the elementary triangle $P\,Q\,S$, formed by the increment of the curve $P\,Q$, the increment of the abscissa $Q\,S$, and the increment of the ordinate $S\,P$, is similar to the triangle $D\,A\,O$, and hence similar to $B\,A\,D$. $\therefore \frac{P\,S}{P\,Q}=\frac{B\,D}{B\,A}=\frac{\sqrt{B\,O}}{\sqrt{B\,A}}=\frac{\sqrt{P\,F}}{\sqrt{B\,A}}$, and $\frac{P\,S}{P\,Q\sqrt{P\,F}}=\frac{1}{\sqrt{B\,A}}$. In like manner, $\frac{Q\,T}{Q\,R\sqrt{Q\,G}}=\frac{1}{\sqrt{B\,A}}$. Hence, $\frac{P\,S}{P\,Q\sqrt{P\,F}}=\frac{Q\,T}{Q\,R\sqrt{Q\,G}}$.

Problem. *To find the path of quickest descent from one point to another not in the same vertical line, the particle being urged by the action of gravity, or of a uniformly accelerating force acting in a direction parallel to a given line.*

We will suppose the descent effected in a non-resisting medium, in the vertical plane passing through the two points; and we must seek the relation between the velocity at any point, and the increments of the curve, and of one of the coördinates of that point.

Let $H$ and $I$, Fig. 2, be the two points, and $H\,P\,Q\,R\,I$ the curve of quickest descent from $H$ to $I$; then, necessarily, the portion of the curve between any two points, as $P$ and $Q$, or $Q$ and $R$, will be the path of quickest descent between those points $P$ and $Q$, or $Q$ and $R$, in the curve; otherwise the whole curve $H\,I$ could not be the path of quickest descent from $H$ to $I$, for the velocity acquired on reaching $R$, and the remaining time through $R\,I$, which depends on this velocity, will be precisely the same by whatever path $R$ be reached; so that the property of the whole curve belongs to every part of it.

Fig. 2.

Let $P\,Q\,R$ be an elementary portion of the curve; then, since the time is less through $P\,Q\,R$ than through any other paths $P\,K\,R$, $P\,N\,R$ on either side, and is greater the further these paths recede

from $P\,Q\,R$, it follows that there must be pairs of patlis, one member on each side of $P\,Q\,R$, in which the times of descent, though each longer than in $P\,Q\,R$, are yet precisely equal. Let $P\,K\,R$ and $P\,N\,R$ be one such pair of paths, each member of the pair deviating infinitely little from $P\,Q\,R$, and divided into two portions at $K$ and $N$ by the horizontal line $L\,T$, drawn so as to cut the curve into two equal (or unequal) portions at $Q$. These elementary portions may be regarded as straight lines described uniformly, $P\,K$ and $P\,N$ with the velocity acquired at $P$, $K\,R$ and $N\,R$ with the velocity acquired at $K$ and $N$. By Proposition 2, the velocity at $P$ is proportional to $\sqrt{P\,F}$, that at $K$ and $N$ to $\sqrt{Q\,G}$; hence, since in uniform motion, time $=\frac{\text{space}}{\text{velocity}}$, and since the sum of the times through $P\,N$, $N\,R$ is equal to the sum of the times through $P\,K$ and $K\,R$, we have

$$\frac{P\,N}{\sqrt{P\,F}}+\frac{N\,R}{\sqrt{Q\,G}}=\frac{P\,K}{\sqrt{P\,F}}+\frac{K\,R}{\sqrt{Q\,G}},\text{ or }\frac{P\,N-P\,K}{\sqrt{P\,F}}=\frac{K\,R-N\,R}{\sqrt{Q\,G}}.$$

Now, to introduce the increments of one of the coördinates of the curve, multiply the numerator and denominator of each fraction by the sum of the quantities whose difference forms its numerator, and then, by Proposition 1, for the product of the sum of the hypothenuses of the elementary triangles $P\,K\,L$, $P\,N\,L$, $K\,R\,T$, $N\,R\,T$, substitute the product of the sum and difference of their bases, and we have $\frac{(L\,N+L\,K)\,K\,N}{(P\,N+P\,K)\sqrt{P\,F}}=\frac{(K\,T+N\,T)\,K\,N}{(K\,R+N\,R)\sqrt{Q\,G}}$; dividing by $N\,K$, then $\frac{L\,N+L\,K}{(P\,N+P\,K)\sqrt{P\,F}}=\frac{K\,T+N\,T}{(K\,R+N\,R)\sqrt{Q\,G}}$.

This equation holds good for every such pair of paths, however near to $P\,Q\,R$, and is true when $K$ and $N$ coincide with $Q$; in that case $L\,N+L\,K=2\,L\,Q=2\,P\,S$, $P\,N+P\,K=2\,P\,Q$, $K\,T+N\,T=2\,Q\,T$, and $K\,R+N\,R=2\,Q\,R$. Hence, substituting these quantities for their equals and dividing by 2, we have,

finally $\frac{PS}{PQ\sqrt{PF}} = \frac{QT}{QR\sqrt{QG}}$, which is precisely the property of the cycloid proved in Proposition 3. So that an arc of a cycloid is the brachistochrone, or curve of quickest descent, under the supposed conditions.

The points $F$ and $G$ in Fig. 1 are in the base of the cycloid, so that, in Fig. 2, *the base of the cycloid must pass through the point from which the descent begins;* this condition is often not distinctly mentioned. If the two points $H$ and $I$ be nearly on a level, the arc may be almost the whole cycloid.

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## THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 216.]

### SECTION III.

#### ON THE MOTIONS AND FIGURE OF A SMALL CIRCULAR PORTION OF FLUID ON THE EARTH'S SURFACE.

24. We shall, in this case, suppose that $\alpha$ is a function of the distance from the centre of the fluid. It will be more convenient, therefore, to express our general equations (20) in terms of other polar coördinates, of which the pole $P$, Fig. 2, does not correspond with the pole of the earth. Regarding the earth as a perfect sphere, let $\psi$ be the distance in arc of the new pole $P$ from the pole of the earth; also let

Fig. 2

$\varrho$ be the distance in arc from the pole $P$,

$\mu$ the angle $SPO$ between $\varrho$ and the meridian,
$\varepsilon$ the alternate angle $NOP$.

If in equations (20) we put $n = 0$, they become the equations of horizontal motions in the case in which the earth has no rotary motion, and the pole of the coördinates can in this case be assumed at pleasure. Hence, when the earth has no rotation, by putting $\varrho$ for $\theta$, and $\mu$ for $\varphi$, we have

$$(33) \quad \begin{aligned} g\, D_\rho h &= r^2 \sin\varrho \cos\varrho\, (D_t \varphi)^2 - r^2 D_t^2 \varrho - g\, h\, D_\rho \log \alpha, \\ g\, D_\mu h &= -2\, r^2 \sin\varrho \cos\varrho\, D_t \varrho\, D_t \mu - r^2 \sin^2 \varrho\, D_t^2 \mu. \end{aligned}$$

When the earth has a rotation, we must add to the second members of these equations respectively the terms $D_\rho F$, and $D_\mu F$, in which $F$ is the part of $P$, equation (10) depending upon the earth's rotation, and must satisfy the following equations,

$$D_\theta F = 2\, r^2 n \sin\theta \cos\theta\, D_t \varphi,$$
$$D_\phi F = -2\, r^2 n \sin\theta \cos\theta\, D_t \theta.$$

Since $\theta$ and $\varphi$ are functions of $\varrho$ and $\mu$, we must put

$$D_\rho F = D_\theta F . D_\rho \theta + D_\phi F . D_\rho \varphi,$$
$$D_\mu F = D_\theta F . D_\mu \theta + D_\phi F . D_\mu \varphi.$$

Hence, substituting the preceding values of $D_\theta F$ and $D_\phi F$, we get

$$(34) \quad \begin{aligned} D_\rho F &= 2\, r^2 n \sin\theta \cos\theta\, (D_t \varphi\, D_\rho \theta - D_t \theta . D_\rho \varphi), \\ D_\mu F &= 2\, r^2 n \sin\theta \cos\theta\, (D_t \varphi\, D_\mu \theta - D_t \theta . D_\mu \varphi). \end{aligned}$$

Now, from the relations of the different parts of a spherical triangle, we have

$$(35) \quad \begin{aligned} \cos\theta &= \cos\psi \cos\varrho - \sin\psi \sin\varrho \cos\mu, \\ \cot\varphi &= \frac{\sin\psi \cos\varrho + \cos\psi \sin\varrho \cos\mu}{\sin\psi \sin\mu}. \end{aligned}$$

Hence, taking the derivatives and reducing, we get

$$D_\rho \theta = \frac{\cos\psi \sin\varrho + \sin\psi \cos\varrho \cos\mu}{\sin\theta} = \cos\varepsilon,$$

$$D_\mu \theta = -\frac{\sin\psi \sin\varrho \sin\mu}{\sin\theta} = -\sin\varrho \sin\varepsilon,$$

$$D_\rho \varphi = \frac{\sin^2\varphi}{\sin\psi \sin\mu} = \frac{\sin\varepsilon}{\sin\theta},$$

$$D_\mu \varphi = \frac{\cos\psi \sin\varrho + \sin\psi \cos\varrho \cos\mu}{\sin^2\mu \sin\varrho} \sin^2\varphi = \frac{\sin\varphi \cos\varepsilon}{\sin\theta},$$

$$D_t \theta = D_\rho \theta \,.\, D_t \varrho + D_\mu \theta \,.\, D_t \mu = \cos\varepsilon\, D_t \varrho - \sin\varrho \sin\varepsilon\, D_t \mu,$$

$$D_t \varphi = D_\rho \varphi \,.\, D_t \varrho + D_\mu \varphi \,.\, D_t \mu = \frac{\sin\varepsilon}{\sin\theta} D_t \varrho + \frac{\sin\varrho \cos\varepsilon}{\sin\theta} D_t \mu.$$

These values being substituted in (34), we get

$$(36) \qquad \begin{aligned} D_\rho F &= 2\, r^2 n \sin\varrho \cos\theta\, D_t \mu, \\ D_\mu F &= -2\, r^2 n \sin\varrho \cos\theta\, D_t \varrho. \end{aligned}$$

If we add these values of $D_\rho F$ and $D_\mu F$ respectively to the second members of (33), we get for the equations of motion, in terms of $\varrho$ and $\mu$, when the earth has a rotation,

$$(37) \qquad \begin{aligned} g D_\rho h &= r^2 \sin\varrho\,(2 n \cos\theta + D_t \mu \cos\varrho)\, D_t \mu - r^2 D_t^2 \varrho - g h D_\rho \log\alpha, \\ g D_\mu h &= -2\, r^2 \sin\varrho\,(n \cos\theta + D_t \mu \cos\varrho)\, D_t \varrho - r^2 \sin^2\varrho\, D_t^2 \mu, \end{aligned}$$

in which $\cos\theta$ has the value in terms of $\varrho$ and $\mu$, in the first of (35).

25. When $\sin\varrho$ is so small that the last term of the value of $\cos\theta$ may be neglected in comparison with the first, we have $\cos\theta = \cos\psi \cos\varrho$, which being substituted in the last equations, they become

$$(38) \qquad \begin{aligned} g D_\rho h &= r^2 \sin\varrho \cos\varrho\,(2 n \cos\psi + D_t \mu)\, D_t \mu - r^2 D_t^2 \varrho - g h D_\rho \log\alpha, \\ g D_\mu h &= -2\, r^2 \sin\varrho \cos\varrho\,(n \cos\psi + D_t \mu)\, D_t \varrho - r^2 \sin^2\varrho\, D_t^2 \mu. \end{aligned}$$

These equations are similar to equations (20), having $\varrho$ and $\mu$ instead of $\theta$ and $\varphi$, and, instead of $n$, having $n \cos\psi$, which is the earth's angular velocity of rotation around the axis, corresponding with the pole $P$ (PEIRCE's *Analytical Mechanics*, § 25). Hence we can treat these equations precisely as equations (20) in the last section, and, instead of (21), we get

(39) $$r^2 \sin^2 \varrho \, (n \cos \psi + D_t \mu) = c,$$

and, instead of (22), we get

(40) $$\int_m r^2 \sin^2 \varrho \, (n \cos \psi + D_t \mu) = \int_m c = Cm.$$

On account of the term which has been neglected in the value of $\cos \theta$, these equations cannot be used for large values of $\varrho$, and hence we may put $\sin \varrho = \varrho$. Let

$s = R \varrho$ be the lineal distance from the centre,

$s'$ be the value of $s$ at the external part of the fluid,

$u$ be the initial value of $D_t \mu$.

The last equation then gives, putting $R$ for $r$,

$$\begin{aligned} Cm &= \int_m s^2 (n \cos \psi + u), \\ &= \int_0^l {}_N \int_0^{2\pi} {}_\mu \int_0^{s'} {}_s \, k \, s^3 (n \cos \psi + u), \\ &= \tfrac{1}{2} s'^2 m \, (n \cos \psi + u'), \end{aligned}$$

in which

$$u' = \frac{2}{s'^2 m} \int_0^l {}_N \int_0^{2\pi} {}_\mu \int_0^{s'} {}_s \, k \, s^3 u.$$

Hence,

(41) $$C = \tfrac{1}{2} s'^2 (n \cos \psi + u').$$

In the preceding integration $k$ is regarded as a constant. When, by the mutual action of the different strata upon each another, $D_t \mu$ becomes the same at all altitudes at the same distance from the centre $P$, $c$ becomes equal to $C$, and equation (39) then gives

(42) $$D_t \mu = \frac{C}{R^2 \sin^2 \varrho} - n \cos \psi = \frac{s'^2 (n \cos \psi + u')}{2 s^2} - n \cos \psi.$$

26. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, the last equation becomes

(43) $$D_t \mu = \left( \frac{s'^2}{2 s^2} - 1 \right) n \cos \psi.$$

Substituting this value of $D_t \mu$ in the first of equations (38), it becomes, by putting $R$ for $r$ and $\cos \varrho = 1$,

$$(44) \qquad g\, D_s h = n^2 \cos^2 \psi \left(\frac{s'^4}{4 s^3} - s\right) - D_t^2 s - g\, h\, D_s \log \alpha.$$

This equation is similar to (27), and, like it, can only be satisfied by means of an interchanging motion between the internal and external part of the fluid; and the remarks following that equation in § 15 are also applicable to this.

27. By omitting the last two terms in the preceding equation, as was done in equation (25), (§ 16), we get by integration,

$$2\, g\, h = - n^2 \cos^2 \psi \left(\frac{s'^4}{4 s^2} + s^2\right) + C.$$

Hence, eliminating $C$,

$$(45) \qquad h = h' + \frac{n^2 \cos^2 \psi}{2 g} \left(\tfrac{5}{4} s'^2 - \frac{s'^4}{4 s^2} - s^2\right).$$

Since one of the negative terms in this value of $h$ has $s$ in the denominator, it must become equal 0 towards the centre where $s$ vanishes. Hence *the fluid, however deep it may be at the external part, cannot exist at the centre.*

28. If we put $s_0$ for the value of $s$ where $h = 0$, the last equation gives

$$(46) \qquad 0 = h' + \frac{n^2 \cos^2 \psi}{2 g} \left(\tfrac{5}{4} s'^2 - \frac{s'^4}{s_0^2} - s_0^2\right),$$

from which we obtain $s_0$ for any assumed value of $h'$.

Since $s_0$ is very small, the terms $\frac{5}{4} s'^2$ and $- s_0^2$ may generally be omitted in the last equation, and it then becomes

$$(47) \qquad s_0 = \frac{n \cos \psi\, s'^2}{\sqrt{2\, g\, h'}}.$$

If we put $s_1$ for $s$ where $h$ is a maximum, equation (44) gives, by putting $D_s h = 0$, and neglecting the last two terms,

$$(48) \qquad s_1 = \frac{s'}{\sqrt{2}}.$$

Equation (45) determines the figure of the surface of the fluid, *which is very slightly convex towards the external part, and meets the surface of the earth near the centre* $c$, as represented in Fig. 3.

Fig. 3.

If we assume $h'$, or $a\,b$, Fig. 3, equal 5 miles, and $a\,c = 100$ miles, equation (49) gives $c\,e = 2$ miles nearly.

29. Equation (43) gives the angular velocity of gyration, which must be very great near the centre, where $s$ is small.

Putting $D_t\,\mu = 0$, it gives

$$(49) \qquad s = \frac{s'}{\sqrt{2}} = s_1.$$

Hence, *at the distance of* $s_1$, *which is the distance of the maximum of* h, *there is no gyratory motion.*

In the northern hemisphere, where $\cos\psi$ is positive, if $s < s'$, $D_t\,\mu$ is positive, but if $s > s'$, it is negative. Hence *the inner part of the fluid gyrates from right to left, but the external part from left to right*, as represented in Fig. 4. In the southern hemisphere, where $\cos\psi$ is negative, the gyrations are the reverse.

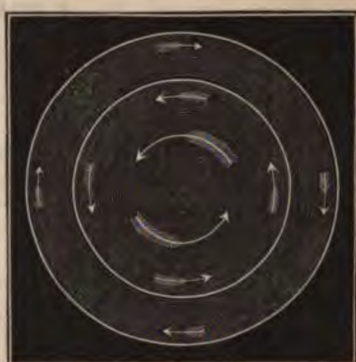
Fig. 4

30. If the fluid is of uniform density, and every part gyrates with the same angular velocity $u$, it satisfies equations (38) by satisfying the following equation:

$$g\,D_s\,h = 2\,s\,u\,n\cos\psi + s\,u^2,$$

since all the other terms vanish; and this motion also satisfies the condition of continuity. By integrating, we get

$$(50) \qquad g\,h = \tfrac{1}{2}\,s^2\,u\,(2\,n\cos\psi + u) + C.$$

This is the equation of a parabola. Hence *the surface of the fluid,*

*relative to the earth's surface, is the surface of a paraboloid.* If the portion of the fluid is so small that the earth's surface may be regarded as a plane, it becomes absolutely the surface of a paraboloid; and when the angular velocity of gyration is great in comparison with that of the earth's rotation, $2n\cos\psi$ may be omitted, in the preceding equation, in connection with $u$.

If $u = -2n\cos\psi$, or $u = 0$, $h$ is constant, and then the surface of the fluid is a level surface. If $u$ is negative and less than $2n\cos\psi$, the surface is convex; in all other cases it is concave.

31. If the whole of a gyrating mass of fluid has a tendency to move in the direction of the meridian with a force $V$, if we regard the forces which act upon each part of the fluid in the directions of the meridians as parallel, we have, using $R$ for $r$,

$$V = m D_t^2 \psi = \frac{1}{R} \int_m D_\theta' P.$$

The error arising from regarding the forces in the directions of the meridians parallel is of the second order of their deviation from parallelism, and consequently very small, unless the lateral extent of the fluid is very great.

From the last equation and the second of equations (9), omitting the term containing $D_t r$ as a factor, since it can produce no sensible effect, we get

$$V = \int_m [-R D_t^2 \theta + R \sin\theta \cos\theta (2n + D_t \varphi) D_t \varphi].$$

If in this equation we substitute for $D_t \varphi$ its value in § 24, and for $D_t^2 \theta$ its value derived from that of $D_t \theta$ in the same section, and also for $\cos\theta$ its value in the first of equations (35), putting $\varepsilon = \mu$, since the meridians are regarded as parallel, and omitting all terms which give 0 by integration, we get

$$(51) \qquad \begin{aligned} R V &= -2n \sin\psi \int_m s^2 \cos^2\mu \, D_t \mu, \\ &= -n \sin\psi \int_m s^2 D_t \mu. \end{aligned}$$

If $D_t \mu$, the angular velocity of gyration, is positive, $V$ is negative; but positive, if $D_t \mu$ is negative. Hence *if the fluid gyrates from right to left, the whole mass has a tendency to move towards the north; but if from left to right, towards the south.*

If every part of a cylindrical mass having its axis of revolution vertical has the same angular velocity of gyration as in the case of solids, calling this velocity $u$, the preceding equation gives for the accelerating force in the direction of the meridian,

$$(52)\qquad \begin{aligned} \frac{V}{m} &= -\frac{s'^2\, u\, n \sin\psi}{2R} = -\frac{s'^2\, u \sin\psi}{2R^2 n} \times R n^2, \\ &= -\frac{s'^2\, u \sin\psi}{2R^2 n} \times \frac{g}{289} = -\frac{g}{578}\cdot\frac{u \sin\psi}{n}\cdot\frac{s'^2}{R^2}. \end{aligned}$$

32. If a body move in the direction of $\varrho$ or $s$ with a velocity $v = D_t s$, and $p$ be the direction of a perpendicular to it on the left, we obtain from the last of equations (36) for the deflecting force in the direction of $p$, arising from the earth's rotation,

$$(53)\qquad \begin{aligned} D_p F &= \frac{D_\mu F}{R \sin\varrho} = -2 R n \cos\theta\, D_t \varrho, \\ &= -2 n \cos\theta\, D_t s = -\frac{2\cos\theta\, D_t s}{R n} \times R n^2, \\ &= -\frac{2\cos\theta\, D_t s}{R n} \times \frac{g}{289} = -\frac{2 g v \cos\theta}{289\, R n}. \end{aligned}$$

This force is negative in the northern hemisphere, and positive in the southern. Hence *in whatever direction a body moves on the surface of the earth, there is a force arising from the earth's rotation, which deflects it to the right in the northern hemisphere, but to the left in the southern.* This is an extension of the principle upon which the theory of the trade winds is based, and which has been heretofore supposed to be true only of bodies moving in the direction of the meridian.

## RESEARCHES IN THE MATHEMATICAL THEORY OF MUSIC.

By Truman Henry Safford, Cambridge, Mass.

1. Before entering directly upon the subject, it may not be inexpedient to premise some things concerning a few of the books upon it. The first writer whose works I have seen is Claudius Ptolemæus, the astronomer of the second century. His work "Harmonics"* is a treatise of the musical scale; what we, in these days, should rather call a work on the theory of Melody. Parts of it seem quite applicable to the music of the present time, and part, to be theoretical only in the sense of an exposition of an untrue hypothesis. It is quite noticeable in the book, that the author uses the same argument to disprove what are now well-known facts that is often now urged against some views which are already held by many musicians, and which I may endeavor to support by mathematical reasoning. Ptolemy denies that the delicacy of the human ear is such that it can distinguish certain variations of tone; he does not pay any regard to those musicians (the disciples of Aristoxenus) who controverted his theory by the evidence of their senses. He evidently had not such an ear for music as his opponents; and his numerical system, if followed out, would have led to intolerable discords.

Aristoxenus seems to have held much the same view that is now entertained concerning the musical scale. Ptolemæus, on the contrary, maintained that the interval of a fourth (as we call it now) must be divided into separate steps, by any "super-particular ratios." That is, any system of three mixed numbers, each of the form $1+\frac{1}{n}$, ($n$ being a whole number) whose product should be $\frac{4}{3}$, would

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* *Κλαυδίου Πτολεμαίου ἁρμονικῶν βιβλία γ′.*

represent in its component parts separate musical intervals, into which a "fourth" could be divided. That this system is utterly opposed to music is known to all conversant with the subject.

The third volume of "WALLIS's Opera Mathematica"* contains PTOLEMY's book, and those of two other writers, much later, — all in Greek, with a Latin translation by WALLIS. A good deal is said in them about the "church tones," scales differing in some respects from our modern major and minor, and professedly lineal descendants of the Dorian, Phrygian, &c., modes of the Greeks.

All these throw light on our modern musical system only incidentally and, as it were, by contrast; but there is one thing mentioned in them theoretically, which actually occurs in modern practice. It is evident, that, if we raise $\frac{2}{3}$ to the third power, we shall get $\frac{8}{27}$, quite a near approximation to $\frac{3}{10}$. Now, in tuning a violin, the fourth string — which should, theoretically, give the note vibrating in $\frac{3}{10}$ the time occupied by that produced by the lowest string (both being open, as it is termed, neither being pressed with the finger while the bow is drawn across them) — really is tuned so as to give the one vibrating in $\frac{8}{27}$ the time of the lowest; and this is effected by the same process represented arithmetically above. The lowest $g$ string is made the foundation; the next or $\bar{d}$ string is made to vibrate in $\frac{2}{3}$ the time of that; the $\bar{a}$ string, again, in $\frac{2}{3}$ the time of that; and, finally, the $\bar{\bar{e}}$ string, or highest, in $\frac{2}{3}$ the time of the latter.

The old writers used theoretically this mode of procedure in all cases; that is, the fraction $\frac{16}{27}$ takes the place of $\frac{3}{5}$, and $\frac{64}{81}$ the place of $\frac{4}{5}$, in their respective harmonic relations. The pure or nearly pure *major third*, indispensable in our day, was ignored entirely.

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* *Πορφυρίου εἰς τὰ ἁρμονικὰ Πτολεμαίου ὑπόμνημα. Μανουὴλ Βρυεννίου ἁρμονικά.*

An interesting work of the last century is one by MARPURG,* a distinguished musician. This is a little book on the Theory of Temperament (a subject relating to the means employed to modify the fractions $\frac{4}{5}$ and $\frac{64}{81}$, or the like, so that, practically, they shall be equal, — which indeed is the great difficulty in the whole matter.) MARPURG, however, writes for people who do not apparently know how to extract roots, and have not seen a table of logarithms. For his treatment of numbers is laborious in the extreme, and he has to teach arithmetic as he goes along. A little knowledge of mathematics would have saved half the pages in his book.

The great EULER,† who has anticipated many things supposed to be late discoveries, seems to have been the first to remark, that musical intervals had something logarithmic in their nature. For as an interval (the difference of pitch of two tones) depends only upon the ratio of frequency of their vibrations; as an interval is added to another interval by multiplying the ratio of vibrations corresponding to one with that corresponding to the other; and as, in consequence, intervals are multiplied and divided by raising to powers and extracting roots of their respective ratios, — we see at once that an interval is best represented by the logarithm of the ratio of times of vibrations of the two notes comprising it.

As, however, the unit of musical intervals is the *octave*, whose times of vibration are as 1 to 2, the logarithms whose base is 2 are the fundamentals.

Much of EULER's "Tentamen" is, however, rather impracticable — too speculative. Were his authority in such matters not so great,

* Anfangsgründe der Theoretischen Musik.

† Tentamen novæ theoriæ Musicæ de certissimis Harmonicæ Principiis dilucide expositæ actore LEONHARDO EULER. Petropoli, ex typographia Academiæ Scientiarum, 1739.

it might perhaps be said, that he had missed the point of the whole subject.

SMITH's Harmonics,* a book published, about a century ago, by a Cambridge mathematician and theologian, is very valuable even at the present day; and may be considered as an indispensable help in the study of a portion of our subject.

Three articles, by Prof. M. W. DROBISCH,† which I have seen, are apparently very good. The difficulties I find with them are, that he almost totally ignores the music of the past age, and holds to that of the present day as the only true development of the art. But one of the greatest German composers — JOHN SEBASTIAN BACH — thought otherwise; and although but of a hundred years' standing, and just beginning to be appreciated, he yet wrote much music in the ancient "church tones."

DROBISCH's other fault seems to be, an attempt to apply "Least Squares" without consideration of "weights." This problem is presented to him: "Certain notes being required to do double duty, to serve in different and utterly distinct chords, which can only do so by virtue of such relations as that between $\frac{3}{5}$ and $\frac{16}{27}$, notes being permitted to sound together which are really only approximately concordant, to determine those notes in accordance with the principles of the method of least squares." Now an approximation to the ratio $\frac{4}{5}$ (perhaps .802) will correspond to two

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* Harmonics, or the Philosophy of Musical Sounds. By ROBERT SMITH, D. D., F. R. S., and Master of Trinity College. Second Edition. London, 1759.

† Über die wissenschaftliche Bestimmung der musikalischen Temperatur, von MORITZ WILHELM DROBISCH, Professor, etc. Leipzig. (From Poggendorff's Annalen der Physik und Chemie, 1853, No. 11, pp. 353–388.)

Ueber Musikalische Tonbestimmung und Temperatur, von M. W. DROBISCH. (Saxon Translations, Vol. IV., p. 1, Leipzig, 1855.)

Ueber Musikalische Tonverhältnisse. (Saxon Translations, Vol. V., p. 1.)

notes sounding much more agreeably together, than two notes whose ratios are as nearly $= \frac{2}{3}$, namely, .6686. Indeed, as DROBISCH himself remarks in another place, the ear can distinguish slighter variations from purity of concord in the latter case, than in the former; and he has himself determined the ratio of such variations. This fact furnishes us with the means of determining, or at least acknowledging, the existence of weights (in the least-square sense of the term). But DROBISCH solves the equations which he obtains, putting all his weights $= 1$. More may be said about this in the proper place.

A good idea of the "Church Modes," so called, can be obtained from a clever little book on that subject by CHARLES CHILD SPENCER.*

Recently Prof. DEMORGAN is said to have written a memoir, which I have not yet seen. I presume, from an abstract of its contents, that it relates chiefly to practical matters; is for organ builders, etc.

Mr. H. W. POOLE,† in his Essay on "Perfect Intonation, and the Euharmonic Organ," introduces some novel ideas, and some which will in future be put in practice. The "Euharmonic Organ" of Messrs. ALLEY and POOLE is practically but little used, and may continue so as long as the organ music of the present day is played.

It may become necessary to cite other works than those above mentioned, as this short list is far from a complete one, even of the few books accessible in America.

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* "Concise Explanation of the Church Modes." London, Novello.

† Silliman's Journal, New Series, Vol. IX., pp. 68, 199.

# Mathematical Monthly Notices.

## ON THE STABILITY OF THE MOTIONS OF THE RINGS OF SATURN.

We have just received an Essay, by J. Clerk Maxwell, M. A., Late Fellow of Trinity College, Cambridge, and Professor of Natural Philosophy in the Marischal College and University of Aberdeen, which obtained the "Adams Prize," in the University of Cambridge. The subject for the Prize is stated in the following words: —

"*The Motions of Saturn's Rings.* The Problem may be treated on the supposition that the system of rings is exactly, or very approximately, concentric with Saturn, and symmetrically disposed about the plane of his equator; and different hypotheses may be made respecting the physical constitution of the rings. It may be supposed (1) that they are rigid; (2) that they are fluid, or in part aeriform; (3) that they consist of masses of matter not mutually coherent. The question will be considered to be answered by ascertaining on these hypotheses severally, whether the conditions of mechanical stability are satisfied by the mutual attractions and motions of the Planet and the rings.

"It is desirable that an attempt should also be made to determine on which of the above hypotheses, the appearance both of the bright rings and the recently discovered dark ring, may be most satisfactorily explained; and to indicate any causes to which a change of form, such as is supposed from a comparison of modern with earlier observations to have taken place, may be attributed."

Laplace investigated the motions of circular homogeneous solid rings, and found that they are in a state of instability, and would, therefore, finally fall upon the Planet and be destroyed. "Hence," he says, "it follows, that the separate rings which surround the body of Saturn are *irregular solids*, of unequal widths in the different parts of their circumferences; so that their centres of gravity do not coincide with their centres of figure. These centres of gravity may be considered as so many satellites, which move about the centre of Saturn, at distances depending on the inequalities of the parts of each ring, and with velocities of rotation equal to those of their respective rings." (Bowditch's Translation.)

Mr. Maxwell, in his preface, states the conclusion thus: "If the rings were solid and uniform, their motion would be unstable, and they would be destroyed. But they are not destroyed, and their motion is stable; therefore they are either not uniform or not solid." It will be seen that this latter alternative of this conclusion, although legitimate, was not considered by Laplace; nor was it seriously entertained till the year 1851. In Gould's Astronomical Journal, Vol. II., No. 1, May 2d, 1851, we find an article "On the Rings of Saturn," by Prof. G. P. Bond, Director of the Observatory of Harvard College, in which the author first treats the question of the multiple divisions of the ring historically, and concludes that the various changes which have been observed are most naturally and easily explained upon the supposition that they are fluid.

He then proceeds to show that "there are considerations to be drawn from the state of the forces acting on the rings which favor the hypothesis." On the supposition of a single solid ring, taking the most probable determinations of the necessary data, he finds "that it will be necessary to increase its attractive force by sixty times its probable value, in order to retain its particles on its surface." The next supposition is a single division into two equal rings; and by

giving each ring such time of rotation as will retain particles on its middle from leaving their place, it is found that the resulting widths of the rings are entirely too great. It is then necessary to suppose a larger number of rings, having different times of rotation, but still with intervals sufficiently small to give the requisite amount of reflecting surface; but the same analysis shows that even this condition is insufficient. LAPLACE proved that these rings could not be uniform homogeneous solids; but if they are irregular solids, then their centres of gravity must revolve about the Planet, and if their inequalities are large enough to oppose their tendency to fall upon the body of the Planet, their mutual disturbance must be so great as to render a collision of the rings very probable, if not wholly unavoidable. The foregoing is a brief outline of the argument from which the author infers "that the whole ring is in a fluid state, or at least does not cohere strongly." "Finally," he says, "a fluid ring, symmetrical in its dimensions, is not of necessity in a state of unstable equilibrium with reference either to Saturn or the other rings."

In GOULD'S Journal, Vol. II., No. 3, June 16th, 1851, we find an article by Prof. PEIRCE, "On the constitution of Saturn's Rings," giving the results of his analysis, "based on purely mechanical considerations," of which the following are the condensed statements:—

1. "I maintain, unconditionally, that *there is no conceivable form of irregularity and no combination of irregularities, consistent with an actual ring, which would serve to retain it permanently about the primary, if it were solid.*" In referring to different cases of irregularity, he says, "In any case, the result is essentially the same,—that they will not permanently support the ring; that a solid ring would soon be destroyed; and that *Saturn's* ring must, therefore, be fluid. It consists, in short, of a stream, or rather streams, of a fluid somewhat denser than water, flowing around the Planet."

2. "Even in the case of a fluid ring, the motion of its centre of gravity is not controlled by the primary."

3. "The power which sustains the centre of gravity of *Saturn's* ring is not, then, to be sought in the planet itself, but in his satellites."

4. "It follows, then, that no planet can have a ring, unless it is surrounded by a sufficient number of properly arranged satellites."

Next, GOULD'S Journal, Vol. IV., No. 14, September 5th, 1855, contains the beginning of a paper by Prof. PEIRCE, "On the Adams Prize Problem for 1856," in which the discussion is divided into three cases of solid, fluid, and discontinuous. Under the head of a solid ring, we find the conclusion stated thus: "*The conditions of the permanence of a solid ring are then necessarily subject to an unstable element; and they are therefore unstable, so that the solid ring must be excluded from any physical theory which rests upon a firm basis.*"

In the second hypothesis of a fluid ring, the conclusion is, that "*The fluid ring cannot then be regarded as one of real permanence without the aid of foreign support; although the action of the primary is not positively destructive to this, as it is to the solid ring.*"

We next come to Mr. MAXWELL'S Essay, and shall simply give the results of his analysis:—

1. A RIGID RING. "The result of this theory of a rigid ring shows not only that a perfectly uniform ring cannot revolve permanently about the planet, but that the irregularity of a permanently revolving ring must be a very observable quantity, the distance between the centre of the ring and its centre of gravity being between .8158 and .8279 of the radius. As there is no appearance about the rings justifying a belief in so great an irregularity, the theory of the solidity of the rings becomes very improbable."

2. A RING OF EQUAL SATELLITES. "We next examined the motion of a ring of equal satellites, and found that if the mass of the planet is sufficient, any disturbances produced in

the arrangement of the ring will be propagated round it in the form of waves, and will not introduce dangerous confusion. If the satellites are unequal, the propagation of the waves will no longer be regular, but disturbances of the ring will in this, as in the former case, produce only waves, and not growing confusion. Supposing the ring to consist, not of a single row of large satellites, but of a cloud of evenly distributed unconnected particles, we found that such a cloud must have a very small density in order to be permanent, and that this is inconsistent with its outer and inner parts moving with the same angular velocity. Supposing the ring to be fluid and continuous, we found that it will be necessarily broken up into small portions.

"We conclude, therefore, that the rings must consist of disconnected particles; these may be either solid or liquid, but they must be independent. The entire system of rings must therefore consist either of a series of many concentric rings, each moving with its own velocity, and having its own system of waves, or else of a confused multitude of revolving particles, not arranged in rings, and continually coming into collision with each other."

It is true, that LAPLACE, in his investigations "On the figure of the Rings of Saturn," after showing, by the same kind of reasoning which he has already used in determining the figures of the Earth and Jupiter, that its transverse sections must be ellipses, in order that "an infinitely thin stratum of fluid, spread upon the surface of the ring, would be in equilibrium by means of the forces acting upon it," makes the hypothesis of a homogeneous fluid ring, and shows that the form of its sections must also be elliptical, in order that particles may be retained on its surface. He also shows that its time of rotation is the same as that of a satellite which should move in the path described by the centre of the generating ellipse of the ring, which ellipse must "vary in magnitude and position throughout the whole extent of the generating circumference of the ring, as such inequalities are necessary to maintain the ring in its equilibrium about Saturn." But from this it is by no means to be inferred that LAPLACE supposed that they are really fluid, or that there is, in the nature of the case, any necessity for such an hypothesis. For he distinctly says, "that the smallness of the width and thickness of any one of the rings, in comparison with its distance from the centre of Saturn, seems to increase the accuracy of the application of the preceding theory to the figure of the ring; and to render more probable the explanation we have given of the manner in which it can be sustained about the planet, by the laws of the equilibrium of fluids."

We have deemed the above exposition of LAPLACE's views necessary, as some, from the perusal of the very able and valuable work on the "History of Physical Astronomy," by ROBERT GRANT, Esq., have supposed that LAPLACE was the first to perceive the necessity of the hypothesis of fluidity.

We have now laid before our readers the results arrived at by those who have most carefully studied this deeply interesting subject. It will be observed that Mr. BOND's argument is based upon the fact, that the conditions of stability of a single solid homogeneous ring of uniform dimensions, or a number of such concentric rings, are inconsistent with the observed dimensions of the ring, and that if they are supposed to be irregular solids, then they must almost inevitably destroy each other. Professors PEIRCE and MAXWELL, however, base their argument upon the mechanical conditions involved in such a system. We have stated the results at which they have arrived with sufficient fulness to show in what respect they agree, and to what extent they differ.

*On* HANSEN'S *Lunar Theory.* By A. CAYLEY, Esq, F. R. S., Quarterly Mathematical Journal, Vol. I., pp. 112–125. (1855.)

*A Memoir on the Problem of Disturbed Elliptic Motion.* By A. CAYLEY, Esq., F. R. S. Read March 9, 1858, before the *Royal Astronomical Society.*

*On the Development of the Disturbing Function in the Lunar Theory.* By A. CAYLEY, Esq., F. R. S. Read November 12, 1858, before the *Royal Astronomical Society.*

We have brought these papers together, because they have the same general aim, namely, the elucidation and more systematic development of HANSEN'S Lunar Theory, as given in his *Fundamenta Nova;* and because the whole subject is now in a much more intelligible and accessible form. The whole argument, as given in these papers, is full and complete; and the reader, with the proper knowledge of analysis, will have no difficulty in understanding it. It will be remembered, that, in the problem of disturbed elliptic motion, the longitudes are measured on the varying plane of the orbit; and the position of this varying plane is determined by reference to the varying plane of the Sun's orbit, and one of the serious difficulties of the problem has been, to take these variations into account; and indeed we do not think that the author states the case too strongly, when he says, "that in memoirs and works on the Lunar and Planetary Theories, it is often difficult to discover where or how (or whether at all) account is taken of these variations." HANSEN has treated the problem rigorously. His system of differential equations involves seven arbitrary constants; six of them define the position of the body with reference to fixed axes, and the seventh determines the position of the arbitrary origin in the orbit from which the longitudes in orbit are counted.

This origin, Mr. CAYLEY calls the "Departure-point;" and longitudes in orbit counted from this point, he calls "Departures." If the node be referred to the departure-point, the seventh constant may be taken as the departure of the node. Now the problem of disturbed elliptic motion is to find the variations of the arbitrary constants in terms of the partial derivatives of the disturbing function, and the departure of the node becomes variable like the other elements.

The departure-point in this case is no longer fixed; but its locus is an orthogonal trajectory of the successive positions of the plane of the orbit; that is, the variation of the longitude of the node projected on the plane of the orbit gives the variation of the departure of the node.

The longitude of the node, the departure of the node, and the inclination, form a group of three elements which fix the position of the plane of the orbit and the departure-point; and two additional elements, the radius vector and departure, give the position of the planet in its orbit. But these last two may be expressed in terms of four others, the mean distance, the mean anomaly, the eccentricity, and the departure of the pericentre. We have then a group of five elements, or of seven; and we may suppose these elements to vary with reference to a fixed plane and a fixed origin of longitudes in this plane; or this plane of reference and origin of longitudes may both be variable, which will give rise to a new source of variation in the first three of these groups of elements. The author has given the formulæ for the variations of these groups of elements when the orbit is referred to a fixed plane, and also when it is referred to a variable plane. When the plane of reference is variable, the origin of longitudes, or departure-point, in this plane, is determined in precisely the same way that it is in the variable plane of the orbit; that is, by the intersections of an orthogonal trajectory of all its successive positions.

After giving the formulæ for the variations of the arbitrary constants, the author proceeds in his next memoir, to the development of the perturbative function, which he effects by HANSEN'S method. All the terms of the development are given in an explicit form, and then compared with the developments of LUBBOCK and PONTECOULANT. Only a single inaccuracy was

found in HANSEN's development. A term in the evection should be $-\frac{3}{2}e+\frac{23}{24}e^3+$ &c., and not $-\frac{3}{2}e+\frac{33}{24}e^3+$ &c., as HANSEN has it. It is desirable to determine to what extent this error affects the moon's place as given by HANSEN's tables.

The form in which the coefficients of the different terms of the development, is given, is admirable on every account. It is better for taking the partial derivatives; and for the numerical reductions, the advantage is equally decided. We regret that want of space prevents our giving a fuller digest of these truly valuable papers, which we heartily commend to the attention of our readers.

*A New Method for Correcting a Planet's Orbit.* By TRUMAN HENRY SAFFORD, A. B. (Published in the Memoirs of the American Academy of Arts and Sciences, Vol. VI. New Series.)

The method here given is somewhat similar to the one in "Theoria Motus," Art. 188; the difference being, that GAUSS's method is one of false position, and Mr. SAFFORD's is differential in its character. The advantage of the differential method is, that it not only shortens the calculations, but enables the computer to dispense altogether with seven figure logarithms, except, perhaps, when the series of observations used possesses uncommon accuracy. Large portions of the computations are made with four and five figure logarithms. The application of the differential method for correcting the orbit is not new; but, as heretofore used, does not seem to have materially shortened the labor. In the above method, however, length of computation is avoided, first by a new set of geocentric coördinates, longitudes and latitudes, referred not to the plane of the Earth's orbit, but to the plane of the approximate orbit already known, of the planet in question; second, by the use of a set of heliocentric differential coördinates, as they may be called, analogous to those which Prof. HANSEN has employed so successfully in the theory of perturbations. This memoir shows that HANSEN's method of representing the effects of small variations in the elements of the orbit upon the heliocentric place is applicable not only to the problem of disturbed elliptic motion, but also to that of pure elliptic motion.

*Researches in the Higher Algebra:* A Paper by JAMES COCKLE, M.A., F. R. A. S.; and read by Rev. ROBERT HARLEY, F. R. A. S., October 5th, 1858, before the Literary and Philosophical Society of Manchester, England.

The following abstract has been communicated to the Monthly:—

"The author, after adverting to the complexity of the results of the higher algebra, proceeds to simplify some of them. For this purpose he employs a set of canonical functions of the unreal fifth roots of unity, and a certain system of six-valued functions of the roots of an equation of the fifth degree. Availing himself of one of the trinomial forms to which Mr. JERRARD and Sir W. R. HAMILTON have shown that the general quintic may be reduced, he has, by an indirect process, succeeded in obtaining the actual expression for the equation of the sixth degree to which that system leads. The resulting sextic is of a simple, and, viewed by the light of Mr. JERRARD's dicoveries, of a comparatively general form. So that the paper may be considered as presenting, on the one hand, the type of a class of equations of the sixth degree, whose finite algebraic solution may be effected by means of one of the fifth, or, on the other hand, as offering a resultant of the sixth degree, the simplicity of which may remove obstacles to the discussion of its solvability. Under the latter aspect the author suggests that his final sextic may perhaps throw light upon the question of the solvability of others, which occur in the theory of quintics."

"In a postscript to the above paper, dated September 10th, 1858, the author indicates the paths which may be pursued in ulterior investigations. He states that Mr. HARLEY, in some as

yet unpublished labors, has verified several of the coefficients of the equation in $\theta$, and introduced improvements into the general theory."

"In a second postscript, dated September 22d, 1858, the author points out that the general solution of a given equation of the fifth degree may be made to depend upon that of the equation in $\theta$."

*The Method of Symmetric Products, and its Application to the Finite Algebraic Solution of Equations:* A Paper by Rev. ROBERT HARLEY, F. R. A. S., read April 5th, 1859, before the Literary and Philosophical Society of Manchester, England.

The following abstract has been communicated to the Monthly:—

"This Paper is divided into three sections. The first contains a systematic exposition of Mr. COCKLE'S Method of Symmetric Products, with illustrations of its power and efficiency when applied to the lower equations. In the second, the Author discusses the resolvent product ($\theta$) for quintics, and defines a new cyclical symbol ($\Sigma'$). He shows that $\theta$ has six, and only six, values, and that, when any one of these values vanishes, the equation of the fifth degree admits of finite algebraic solution: its roots are actually exhibited. Mr. COCKLE'S new solvable form is verified, and shown to include, as particular cases, the quadrinomials of DE MOIVRE and EULER. The third section contains a direct calculation of the equation in $\theta$. The coefficients are followed, one by one; the calculation being carried on by means of the cyclical symbol $\Sigma'$, which is shown to possess peculiar working properties. The resulting sextic is found to coincide with Mr. COCKLE'S equation, obtained by a wholly different method, which was laid before the Society a few months ago in his 'Researches in the Higher Algebra.' The author notices the steadiness with which the Method of Symmetric Products mounts up to the higher equations, and concludes by expressing his belief that the equation in $\theta$—the verification of which has involved prodigious labor—will be found to be a canonical equation in the theory of quintics."

*Rational Cosmology: or the Eternal Principles and the Necessary Laws of the Universe.* By LAURENS P. HICKOK, D. D., Union College, Schenectady, N. Y. New York: D. Appleton and Co. 1858.

From the last paragraph on page iv. of the first number of the Mathematical Monthly, I infer it to be the intention of the editor occasionally to notice and review such new mathematical publications as may from time to time appear. I would respectfully ask your attention to the above work. The high standing of the author, and the well-merited reputation he has acquired for his previous philosophical productions, are such as cannot fail to insure a favorable reception to this latest and *favorite* (as I am assured) effort of his energetic pen. But as the title of the present work is not mathematical, it is not directly adapted to mathematical readers, and thus might not receive that prompt criticism and exposure which its interference deserves. With an obviously superficial knowledge of the principles and conditions of demonstration requisite in the treatment of questions of mechanical and physical science, and with an evident foregone conclusion in his mind, and, as is easily obtained from a slight reading of very elementary treatises, our author rushes to the end of his demonstration with a rapidity truly alarming to the cautious reasoner, and apparently unconscious that a *non-sequitur* is grinning at his heels. If his book shall be suffered to remain before the public without the application of a little wholesome mathematical logic by way of a corrective, it may, from the conceded ability and widespread reputation of its author, seconded by the captivating generality and pliant subserviency of the hypothesis which he has made the basis of all his deductions (?), tend to impose for a time upon many merely literary or even *quasi*-scientific readers.

But a greater evil may result, if the errors in detail, of which two thirds of this volume are

made up, are not timely condemned by a mathematical tribunal. If such astonishing oversights are committed in the treatment of mechanical and physical questions, where the data are furnished us from without by the hand of nature, how far is there safety in relying upon the treatment of moral questions, in which the data and the method of reasoning are both at the entire disposition of the reasoner ? His psychological and moral writings have obtained a favorable verdict: Behold now his long-promised theory of physics! Shall this last be allowed to act to the discredit of the former ? No ! it is only another instance in evidence, that the human mind of itself alone, unaided by the coöperation of material forces and the observation of material phenomena, is totally unable to divine the laws of the phenomona of the universe, or to discover the principles that determine the fulfilment of these laws. Just as the substance water is the resultant of the two component forces which manifest themselves separately in the form of oxygen and hydrogen, so man's knowledge is the resultant given by the combination of material with spiritual forces, and not by the exercise alone of spiritual force upon inactive or plastic matter.

From among the numerous examples of inconsequent reasoning which fill the greater part of the pretentious volume in question, the following may be selected as one of sufficient prominence to deserve attention. After arriving, by a very peculiar route, at the conclusion, that the force of gravitation decreases in energy according to the inverse square of the distance from the attracting centre (in which remarkable deduction, by the way, the two different relations of *difference* and *quotient* are confounded in the one term *inverse*), the demonstration of the law of falling bodies is taken up at page 157 of the Cosmology. Under the avowed hypothesis, that the energy of the attractive force increases with the increase of the inverse square of the distance from the centre, the demonstration obviously starts with the supposition that such energy increases uniformly with the decrease of the distance to the centre; while the results reached, as expressed in the uniform increase of the velocity of the falling body and the distance described by the fall, are the results due to the action of a *constant* force, such as is the force of gravitation throughout a small change of distance towards the centre. This conclusion the author was bound to reach, in order to avoid any discrepancy that might arise between the *rational* demonstration and the well-known mathematical and experimental ones; his result is safe, but the haphazard method of reaching it is open to grave objection. The accompanying figure is given to facilitate the application of numerical computation to a few steps of the author's verbal deductions.


$C$ being the attracting centre and $O$ the position of the body destined to describe a distance increasing as the square of the time of its fall, the crosses mark infinitesimal units of distance $1\,l$, and the right-hand figures denote the corresponding energy of the central force. For if, as the Cosmology states, the energy be called 1 at the point $O$, and become 2 at the point so marked, it must increase by unity for each unit of approach towards the centre, and so become 3, 4, 5, &c., at the points thus reached. By virtue of the attracting force $1\,\phi$ at the point $O$, the body falls through the distance $1\,l$ in the first moment of time: at the point 2, the accession of energy is $2\phi$, and, the former energy being retained, the sum $3\phi$ carries three steps to the point 5, during the second moment of time, making the distance $4\,l$ from the origin $O$. So far appears to be plain enough; but now, at the point 5, the accession of energy is evidently $5\,\phi$, and in addition there is on hand also the accessions $3\,\phi$ and $4\,\phi$ received at their respective points, making in all the sum $(5+4+3+2+1)\,\varphi = 15\,\varphi$, with which energy the body must proceed during the third moment of its fall, which is an increase of rapidity not at all contemplated at the outset; since the distance described in the moment should really be $5\,l$, making

the distance $9l$ from the origin at the expiration of that time. True, the demonstrator quotes the number 2 as the measure of the accession of velocity received by the falling body at the end of each successive moment of time; but this flatly contradicts the hypothesis of an increased energy of the force during the fall, and places the question within a province of reasoning, with which the rational cosmologist is obviously but superficially acquainted.

Faults of reasoning and misapprehension of principles equally gross, appear in nearly every example given in this systematic attempt to detail the application of the *absolutely* philosophical method to the constructive demonstration of the facts and phenomena of the natural world. A reference to the discussion of the second and third laws of motion, at pages 122 and 126, may please the mathematical tyro; while physicians and astronomers will doubtless be mutually edified by the author's theories of terrestrial magnetism, and of the formation of the galaxy, &c. &c.

Θ.

## Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the March number of the MONTHLY:—

GEORGE B. HICKS, Student, Cleveland, Ohio, answered questions II., IV., and V.

ASHER B. EVANS, Junior Class, Madison University, Hamilton, N. Y., answered questions I., III., and IV.

If it is the opinion of teachers that the Prize Problems we select are too difficult, we shall be very much obliged if they will send us such as they think suitable. We are anxious not to defeat the desired end by giving problems either too easy or too difficult. . . . . . At a meeting of the Board of Trustees of Columbia College, held the 1st Monday in May, Professor CHARLES DAVIES was appointed Professor of Higher Mathematics, and WILLIAM G. PECK, Professor of Mathematics. . . . . . In the Note on page 279, we referred to two solutions of a problem on tangent circles in GILL'S *Mathematical Miscellany;* but did not at the time notice that it also contains interesting solutions by Professors G. B. DOCHARTY, MARCUS CATLIN, BENJAMIN PEIRCE, C. GILL, and T. STRONG, of other special cases of the problem on the tangency of circles. . . . . . Professor HOYT writes: "In the April No., p. 231, line 11, for 'A. D.' read 'that line.' The perpendicular is not represented in the figure. The mode of division is essentially the same as the second one described on pages 159, 160." It is proper to state that Professor HOYT'S manuscript was received before the issue of the February No., containing the construction to which he refers. . . . . . In the May No., Prize Problem II., for "Transpose" read "Transform;" on page 263, line —3, for "$R$" read "$K$;" on page 282, line —12, for "Art. 1135," read "Art. 1535." . . . . . We must beg of our correspondents to send us plain manuscript, and carefully drawn cuts of the right size; so that they can be put into the hands of the printer and engraver just as we receive them.

THE

# MATHEMATICAL MONTHLY.

Vol. I....JULY, 1859....No. X.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

SOLVE the two equations

$$x^2 - 50\,x + xy + x^2y^5 + xy^6 = 50\,y$$
$$x^2y - 100\,x + xy^2(1+y^2)(1+x) + xy^5 = 100\,y.$$

### II.

If $A$, $B$, $C$ be the angles, and $a$, $b$, $c$ the opposite sides, in a plane triangle, of which $S$ denotes the surface; prove that

$$a^2 + b^2 + c^2 = 4\,S(\cot A + \cot B + \cot C).$$

### III.

If one of the similar triangles $ABC$ and $A'B'C'$ be inscribed in the triangle $DEF$, and the other circumscribed about it; prove that the area of $DEF$ will be a mean proportional between the areas of $ABC$ and $A'B'C'$.

### IV.

If $a$ be one of the sides of an equilateral spherical triangle and $A$ one of its angles, prove that $\sec A = \sec a + 1$.

### V.

If the semiaxes of an ellipse be $A$ and $B$, $p$ the length of the

perpendicular dropped from the centre on the tangent to the curve, $r$ and $r'$ the distances from the point of tangency to the foci, and $\varrho$ the radius of curvature at this point; prove that

$$\varrho = \frac{A^2 B^2}{p^3} = \frac{r r'}{p};$$

and from this theorem construct the corresponding point of the evolute.

The solution of these problems must be received by the first of September, 1859.

## REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VI., Vol. I.

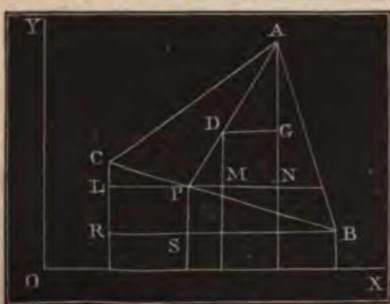
The first Prize is awarded to George B. Hicks, of Cleveland, Ohio.

The second Prize is awarded to Asher B. Evans, of the Junior Class in Madison University, Hamilton, N. Y.

### Prize Solution of Problem I.

"Any side of a triangle is cut in the ratio of $m$ to $n$, and the line joining this point to the opposite vertex is cut in the ratio of $m+n$ to $l$; to find the coördinates of the point of section."

Let $ABC$ be any triangle, and let the coördinates of the vertices $A, B, C$ be $x'y'$, $x''y''$, $x'''y'''$. Let the side $BC$ be cut at




$P$ in the ratio of $m$ to $n$, and let $xy$ be the coördinates of $P$. The triangles $BPS$ and $PCL$ give

$BP:PC::BS:LP$; or, $n:m::x''-x:x-x'''$

$BP:PC::PS:CL$; or, $n:m::y-y'':y'''-y$.

$$\therefore \quad x = \frac{mx''+nx'''}{m+n}, \quad y = \frac{my''+ny'''}{m+n}.$$

Again, let $AP$ be cut at $D$ in the ratio of $m+n$ to $l$, and let

$\alpha, \beta$ denote the coördinates of $D$. The triangles $ADC$ and $DPM$ give

$$AD:DP::DG:PM;\ \text{or,}\ m+n:l::x'-\alpha:\alpha-\frac{mx''+nx'''}{m+n},$$

$$AD:DP::AG:DM;\ \text{or,}\ m+n:l::y'-\beta:\beta-\frac{my''+ny'''}{m+n}.$$

Therefore, $$\alpha=\frac{lx'+mx''+nx'''}{l+m+n},\ \beta=\frac{ly'+my''\,ny'''}{l+m+n},$$

are the required coördinates.

This solution is by George B. Hicks.

PRIZE SOLUTION OF PROBLEM III.

"Find the polar equation of the line passing through the points, of which the polar coördinates are $r', \varphi'$; $r'', \varphi''$."

The equation of the straight line referred to rectangular coördinates, and passing through the points $x'y'$, and $x''y''$, is

$$(1) \qquad y-y'=\frac{y'-y''}{x'-x''}(x-x').$$

If the pole be at the origin, and the axis of $x$ be the polar axis, then $x=r\cos\varphi$ and $y=r\sin\varphi$. Accenting, making the substitutions and reducing, (1) becomes

$$rr'(\sin\varphi\cos\varphi'-\cos\varphi\sin\varphi')+rr''(\sin\varphi''\cos\varphi-\cos\varphi''\sin\varphi)$$
$$+r'r''(\sin\varphi'\cos\varphi''-\cos\varphi'\sin\varphi'')=0;$$

or, $rr'\sin(\varphi-\varphi')+rr''\sin(\varphi''-\varphi)+r'r''\sin(\varphi'-\varphi'')=0$, which is the equation required. This solution is by George B. Hicks.

PRIZE SOLUTION OF PROBLEM IV.

"Find the condition that $Ax+By+C=0$ should be tangent to $(x-a)^2+(y-b)^2=r^2$."

Let $y$ be eliminated between the two equations, and the result arranged according to powers of $x$; and we shall have

$$(A^2+B^2)x^2+2(AC+bAB-aB^2)x+B^2(a^2+b^2)+2bBC+C^2-r^2B^2=0.$$

In order that $Ax+By+C=0$ may be tangent to the circle, the

values of $x$ from this quadratic must be equal. Whence, by the theory of equal roots,

$$(A\,C + b\,A\,B - a\,B^2)^2 = (A^2 + B^2)\,(a^2\,B^2 + b^2\,B^2 + 2\,b\,B\,C + C^2 - r^2\,B^2).$$

Developing and reducing, this equation becomes

$$r^2\,(A^2 + B^2) = (A\,a + B\,b + C)^2,$$

which is the required condition.*

The condition may be verified as follows: It may be written $r = \pm\,\dfrac{A\,a + B\,b + C}{\sqrt{A^2 + B^2}}$. Now, we know by a simple proposition in analytical geometry, that the right-hand member of this equation expresses the length of the perpendicular from the point $a, b$ upon the line $A\,x + B\,y + C = 0$; and since this perpendicular is equal to the radius of the circle, the coördinates of whose centre is $a, b$, the condition of tangency is fulfilled, as is known from plane geometry. It may be remarked also, that this condition gives a very concise solution of the problem. This solution is by GEORGE B. HICKS.

JOSEPH WINLOCK.
CHAUNCEY WRIGHT.
TRUMAN HENRY SAFFORD.

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* The method of determining the equation of a tangent used here is the invention of DESCARTES. It is not confined in its application to curves of the second degree, but is generally applicable to all curves. Let the equation of any curve be $F\,(x\,y) = 0$, and between this and the equation of the right line eliminate either variable, as $y$. The resulting equation will involve only $x$, the roots of which will be the values of $x$ for the points where the line intersects the curve. If two of these points unite, two of the roots will be equal, and the line will become a tangent. The method by which DESCARTES determined the condition under which two of the roots would be equal was by assuming an equation of the same degree having two equal roots, and comparing it with the resulting equation. When the given curve is of the second degree, this ingenious artifice is rendered unnecessary, the solution of the equation being sufficient; as it is only necessary to put the radical term of the roots equal to zero.

This method of determining the equation of a tangent is that which appears in the letters of DESCARTES. That which he gives in his Geometry is somewhat different, and nearly as follows: Let

$$y^2 + (x - x_0)^2 - r^2 = 0$$

be the equation of a circle, the centre of which is on the axis of $x$. Let $y$ be eliminated by means of this equation and that of the curve, and the roots of the resulting equation will be the values of $x$ for the points where the circle meets the curve. The centre of the circle being supposed fixed and the radius arbitrary, let it be supposed to have such a value as will render two of the roots of the equation equal; the circle will then touch the curve, and therefore have the same rectilinear tangent. The value of $r$ which renders the roots equal may be found by the artifice mentioned above. These methods are both founded on the same principle; and though we cannot but admire the ingenuity they display, yet they must in general yield to the more simple and direct method furnished by the Calculus.

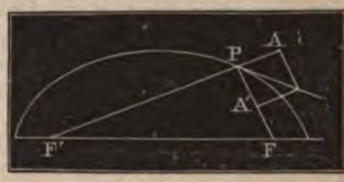
Both of the above methods were used in the Prize solutions in the June number of the MONTHLY. See solutions of Problems I. and III.

The method of drawing tangents to curves, founded on the principles of the Differential Calculus, has superseded the other solutions for the same problem given by DESCARTES, FERMAT, ROBERVAL, and others. The methods given by these geometers were either limited to particular classes of curves, or in some cases so incommodious as to amount nearly to impracticability. The determination of the equation of a tangent by the calculus is at once simple and general. It depends merely on differentiating the equation of the curve to find $\frac{dy}{dx}$, the tangent of the angle which the tangent line makes with the axis of $x$, and therefore extends to every curve capable of being expressed by an equation, and whose equation is capable of differentiation. The methods of DESCARTES, explained above, extend at most only to algebraic curves.

The method of ROBERVAL deserves notice, as well on account of the elegance of the conception on which it is founded, as of its close analogy to the fundamental principle of the Newtonian fluxions. He considered a curve described by a point affected with two motions, the variations of which in quantity and direction are to be determined by the nature of the curve. At any point of the curve he supposed a parallelogram constructed, the sides of which are proportional to and in the direction of the generating velocities, and laid it down as a principle, that the diagonal which represents the direction of the resultant is the direction of the element of the curve at that point, and therefore the direction of the tangent. There are many instances in which this method may be applied with great clearness and facility; but in most cases its application is either totally impracticable, or attended with very perplexing difficulties, owing to the intricacy of the investigations necessary to determine the component velocities of the generating points. We shall give some examples in which its application is effected with great clearness and beauty.

1. To determine the tangent to a point in an ellipse or hyperbola.

In the ellipse the sum of the distances $F' P$ and $F P$ of the describing point from the foci is invariable; therefore one increases with the same velocity as the other diminishes. Hence the velocity of the describing point in the directions $P A$ and $P A'$ are equal; therefore if $P A = P A'$, the diagonal is the tangent which bisects the angle $A P A'$.




In the hyperbola the difference of the distances from the foci is constant, and therefore the two distances increase with the same velocity. Hence $P A'$ should in this case be taken on the produced part of the focal distance, as well as $P A$, and therefore the tangent bisects the angle under the radii vectores from the foci.

2. To draw a tangent to a given point in a parabola.

Let $OB$ be the directrix, and $OX$ the axis, and $F$ the focus. By the properties of this curve, $FP = BP$. $\therefore$ the velocities in the directions $PA$ and $AP'$ are equal; $\therefore$ as before, the tangent bisects the angle $APA'$.


3. To draw a tangent at a given point in a cycloid.

Let $P$ be the generating point. By the definition of the cycloid, the generating point at $P$ has two motions, one in the direction of the tangent $PA'$ to the generating circle, and the other in the direction $PA$ parallel to the base; and these two motions are equal, because the generating point moves uniformly round the circumference of the generating circle in the same time that the circle itself is carried along the base through an equal space. Hence if $PA$ and $PA'$ represent the two motions $PA = PA'$, and therefore the tangent bisects the angle $APA'$, and is parallel to the corresponding chord $P'V$ of the generating circle described upon the axis.


This method of ROBERVAL is peculiarly applicable to curves which can be described mechanically by motion. BARROW subsequently invented a method of tangents which approached as near the principle of the differential calculus as ROBERVAL's did to the fluxional principle. He investigates an infinitely small triangle composed of the increments of the abscissa and ordinate, and the elementary arc of the curve. The student will readily perceive this to be the spirit of the differential calculus; but both this and the method of ROBERVAL want, what constitutes the principal excellence of the methods of the fluxional and differential calculus, that uniform algorithm by which a general formula expresses the equation of a tangent to any curve, and the general rules by which the particular values of the quantities composing this general formula can be found in particular cases. It should be observed, that the method of BARROW is very nearly the same as that of FERMAT. This note is taken from LARDNER's Algebraic Geometry. — ED.

## NOTE ON DERIVATIVES.

IT seems to us, that the method of demonstrating the rules for finding the derivatives of many algebraic functions is not only most concise, but most easily understood by the learner, when based upon the following

PROPOSITION.* When $i$ is an infinitesimal, Nap. log $(1 + i) = i$. By the binominal theorem,

* See Prof. PEIRCE's *Curves and Functions*, Vol. I.; also Prof. PRICE's *Infinitesimal Calculus*, Vol. I.

$$(1+i)^{\frac{1}{i}}=1+\tfrac{1}{i}.i+\tfrac{1}{i}\left(\tfrac{1}{i}-1\right)\tfrac{i^2}{1.2}+\tfrac{1}{i}\left(\tfrac{1}{i}-1\right)\left(\tfrac{1}{i}-2\right)\tfrac{i^3}{1.2.3}+\text{\&c.}$$

Since $\frac{1}{i}$ is infinite, and since the finite quantities which are added to or subtracted from infinite ones do not affect their values, $\frac{1}{i}=\frac{1}{i}-1=\frac{1}{i}-2=$ &c., which reduces the development to

$$(1+i)^{\frac{1}{i}}=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{\&c.}$$

But this series, by reducing, and adding a sufficient number of its terms together, gives 2.7182818+, known as $e$, the base of NAPIER'S system of logarithms. Hence $(1+i)^{\frac{1}{i}}=e$; and

$$\tfrac{1}{i}\,\text{Nap. log}\,(1+i)=\text{Nap. log}\,e=1. \quad \therefore \log(1+i)=i,$$

understanding that log denotes Nap. log.

We will now apply this proposition to finding the derivatives of a few functions, including Problem V., No. VI., Vol. I.

1. Find the derivative of $u=x^n$. $\log u=n\log x$. Give $x$ an infinitesimal increment $dx$, and let $du$ be the corresponding increment of $u$; then $\log(u+du)=n\log(x+dx)$. The logarithm of the ratio of $u+du$ to $u$ is

$$\log(u+du)-\log u=n\log(x+dx)-n\log x;$$

or, $$\log\left(1+\frac{du}{u}\right)=n\log\left(1+\frac{dx}{x}\right).$$

Therefore, since $\frac{du}{u}$ and $\frac{dx}{x}$ are infinitesimals, we have by our proposition, $$\log\left(1+\frac{du}{u}\right)=\frac{du}{u}=n\log\left(1+\frac{dx}{x}\right)=\frac{n\,dx}{x}.$$

Therefore, $$\frac{du}{dx}=\frac{nu}{x}=\frac{nx^n}{x}=nx^{n-1}.$$

In this case $u$ is called a function of $x$; that is, $u$ depends upon $x$ for its value, and varies with it. But it is evident, that $u$ and $x$ do not vary by the same amount; and it is the aim of the Differential Calculus to find the *ratio* of these variations when they are infini-

tesimals. This ratio, or quotient, is called by another name in the calculus; namely, *derivative* or *differential coefficient.* We started with the function $x^n$, and derived $n x^{n-1}$ from it, as the ratio of the variation of the function to its variable $x$; and hence $n x^{n-1}$ is called the *derivative*, or *derived function* of $x^n$. Again, $du = n x^{n-1} dx$, in which $n x^{n-1}$ is the *coefficient* of the differential $dx$; and hence the name, *differential coefficient.* The student will observe, that $\frac{du}{dx}$ denotes the derivative of the quantity $u$; but the symbol, as separated from the quantity, and simply denoting the operation, is $\frac{d}{dx}$. Thus, $\frac{d}{dx}f(x)$ tells us to find the derivative of $f(x)$. The inconvenience of the use of the symbol $\frac{d}{dx}$, in this and like cases has led to the adoption of $D$ in its place. If we wish to indicate at the same time the particular variable, as $x$, in reference to which the derivative is to be taken, then the symbol $D_x$ is used. Hence in symbols $\frac{d}{dx} = D_x$. When the function involves only a single variable, as $x$, $D$ is sufficient; but in the symbol $\frac{d}{dx}$, the $x$ cannot be omitted. In the case of a general function, as $f(x)$, the notation $f'(x), f''(x), f'''(x)$, &c., to denote the successive derivatives, was used by LAGRANGE, and is most convenient.

So far as we know, Prof. PEIRCE is the only author in this country who has used $D$; and we have made these remarks for the benefit of those students who meet with this notation only in the MONTHLY. We do not advise the exclusive use of either notation. Use the one most convenient in the particular case. In all cases, however, in which we simply wish to indicate the operation, $D$ is preferable. Thus $D x^n$ is better than $\frac{d}{dx} x^n$.

2. Find the derivative of $u = a^x$; $\log u = x \log a$. Giving increments $\log(u + du) = (x + dx) \log a$; and taking the ratio,

$$\log(u + du) - \log u = (x + dx) \log a - x \log a;$$

or, $$\log\left(1+\frac{du}{u}\right)=dx\log a.$$

Therefore, by the proposition

$$\log\left(1+\frac{du}{u}\right)=\frac{du}{u}=dx\log a.\ \therefore\ \frac{du}{dx}=u\log a=a^x\log a.$$

3. Find the derivative of $u=ax^n$. $\log u=n\log x+\log a$. $\log(u+du)=n\log(x+dx)+\log a$. Hence

$$\log(u+du)-\log u=n\log(x+dx)-n\log x;$$

or, $$\log\left(1+\frac{du}{u}\right)=n\log\left(1+\frac{dx}{x}\right)\ \therefore\ \frac{du}{u}=\frac{n\,dx}{x};\ \frac{du}{dx}=nax^{n-1}.$$

The constant factor in the function is still in the derivative.

4. Find the derivative of $u=x^{\frac{p}{q}}$. $\log u=\frac{p}{q}\log x$.

$$\log(u+du)=\frac{p}{q}\log(x+dx);\ \log\left(1+\frac{du}{u}\right)=\frac{p}{q}\log\left(1+\frac{dx}{x}\right).$$

$$\therefore\qquad \frac{du}{u}=\frac{p}{q}.\frac{dx}{x};\ \text{or}\ \frac{du}{dx}=\frac{p}{q}.\frac{u}{x}=\frac{p}{q}x^{\frac{p}{q}-1}.$$

Hence the rule for finding the derivative of a power of a variable is precisely the same for all exponents, whether integral or fractional, and, as is readily seen, whether positive or negative.

5. Find the derivative of $u=x^n+C$, $C$ being a constant quantity. $u-C=x^n$; $\log(u-C)=n\log x$. $\log(u-C+du)=n\log(x+dx)$;

$$\log\left(1+\frac{du}{u-C}\right)=n\log\left(1+\frac{dx}{x}\right).\quad \therefore\ \frac{du}{u-C}=\frac{n\,dx}{x};$$

or $$\frac{du}{dx}=\frac{n(u-C)}{x}=nx^{n-1}.$$

Therefore, a constant which is not a factor disappears in the derivative.

6. Find the derivatives of $u=\log x$. $u+du\quad\log(x+dx)$; $du=\log(x+dx)-\log x=\log\left(1+\frac{dx}{x}\right)=\frac{dx}{x}\ \therefore\ \frac{du}{dx}=\frac{1}{x}$.

7. *Functions of more than one variable.* Find the derivative of $u=x^m y^n$. $\log u=m\log x+n\log y$.

$$\log(u+du)=m\log(x+dx)+n\log(y+dy)$$

$$\log\left(1+\frac{du}{u}\right)=m\log\left(1+\frac{dx}{x}\right)+n\log\left(1+\frac{dy}{y}\right).$$

$$\therefore\quad \frac{du}{u}=\frac{m\,dx}{x}+\frac{n\,dy}{y};\ \text{or, } du=m\,x^{m-1}y^n\,dx+n\,y^{n-1}x^m\,dy.$$

If either $x$ or $y$ be made the independent variable, the derivatives are

$$D_x u=mx^{m-1}y^n+n\,y^{n-1}x^m D_x y;\ D_y u=mx^{m-1}y^n D_y x+n\,y^{n-1}x^m.$$

The values of these derivatives, $D_x u$ and $D_y u$, are not so symmetrical in their form as the value of the differential $du$; which is true, in general, of functions of more than one variable. It is therefore, usually better to retain the differential form.

8. Find the differential of the fraction $u=\frac{x}{y}$.

$$\log u=\log x-\log y;\ \log(u+du)=\log(x+dx)-\log(y+dy).$$

$$\log\left(1+\frac{du}{u}\right)=\log\left(1+\frac{dx}{x}\right)-\log\left(1+\frac{dy}{y}\right).\ \therefore\frac{du}{u}=\frac{dx}{x}-\frac{dy}{y}.$$

$$\therefore\qquad du=\frac{u\,dx}{x}-\frac{u\,dy}{y}=\frac{dx}{y}-\frac{x\,dy}{y^2}=\frac{y\,dx-x\,dy}{y^2}.$$

9. Find the differential of $u=y^x$. $\log u=x\log y$.

$$\log(u+du)=(x+dx)\log(y+dy)=x\log(y+dy)+dx\log(y+dy).$$

$$=x\log(y+dy)+dx\log y\left(1+\frac{dy}{y}\right).$$

$$=x\log(y+dy)+dx\log y+dx\log\left(1+\frac{dy}{y}\right).$$

$$\log\left(1+\frac{du}{u}\right)=x\log\left(1+\frac{dy}{y}\right)+dx\log y+dx\log\left(1+\frac{dy}{y}\right).$$

$$\therefore\qquad \frac{du}{u}=\frac{x\,dy}{y}+dx\log y+\frac{dx\,dy}{y}.$$

$$du=\frac{x\,u\,dy}{y}+u\,dx\log y=x\,y^{x-1}dy+y^x\,dx\log y.$$

It will be seen, that $\frac{dx\,dy}{y}$, which is of the second order, is omitted.

## NOTES ON THE THEORY OF PROBABILITIES.

By Simon Newcomb, Nautical Almanac Office, Cambridge.

13. Some theorems relating to combinations will be given here for convenience of reference.

If we have a set of $n$ things, the number of different ways in which a collection of $s$ things can be taken from it, is called the number of combinations of $s$ in $n$, and is represented by the notation $\overset{s}{\underset{n}{C}}$, or $C(s, n-s)$. Since, for every collection of $s$ things which can be taken, there will remain a collection of $n-s$ things not taken, we may consider the number of combinations of $s$ in $n$ as being the number of ways in which a collection of $s$ things can be divided into two sets, one containing $s$ things, the other $n-s$. It is shown in treatises on Algebra, that

$$(1) \quad \overset{s}{\underset{n}{C}} = \frac{n(n-1)(n-2)\ldots.(n-s+1)}{1.2.3\ldots.s} = \frac{n!}{(n-s)!\,s!}.$$

From the above, we may easily deduce the following equations:

$$(2) \quad \overset{s+1}{\underset{s}{C}} = \frac{n-s}{s+1}\overset{s}{\underset{n}{C}}; \quad \overset{s-1}{\underset{n}{C}} = \frac{s}{n-s+1}\overset{s}{\underset{n}{C}}; \quad \overset{s}{\underset{n}{C}} = \overset{n-s}{\underset{n}{C}}; \quad \overset{0}{\underset{n}{C}} = \overset{n}{\underset{n}{C}} = 1.$$

Using the notation of combinations, the binominal theorem may be expressed in the form

$$(3) \quad (a+x)^n = \overset{0}{\underset{n}{C}}a^n + \overset{1}{\underset{n}{C}}a^{n-1}x + \overset{2}{\underset{n}{C}}a^{n-2}x^2 + \ldots = \sum_0^\infty{}_s \overset{s}{\underset{n}{C}}a^{n-s}x^s.$$

Suppose that the set is divided into two classes, white and black, for example; then a collection of $s$ things may be composed of any $s$ white things,

| | | | | |
|---|---|---|---|---|
| or any | $s-1$ | white combined with any | 1 | black, |
| " | $s-2$ | " " | 2 | " |
| " | &c. | " " | &c. | " |
| " | 1 | " " | $s-1$ | " |
| " | 0 | " " | $s$ | " |

from which we deduce the general theorem

$$(4) \qquad \overset{s}{\underset{n}{C}} = \overset{s}{\underset{n-l}{C}} + l\overset{s-1}{\underset{n-l}{C}} + \overset{2}{\underset{l}{C}}\overset{s-2}{\underset{n-l}{C}} + \text{\&c.} \ldots + \overset{s-1}{\underset{l}{C}}\overset{1}{\underset{n-l}{C}} + \overset{s}{\underset{l}{C}} = \Sigma_r \overset{r}{\underset{l}{C}}\overset{s-r}{\underset{n-l}{C}},$$

where $l$ may be any number (at pleasure) less than $n$.

If we represent the number of ways in which a number of things equal to $(s+s'+s''+\ldots.\text{\&c.})$ can be divided into sets, of which one shall contain $s$ things, another $s'$, &c., by $C(s, s', s''\ldots)$, we have

$$C(s, s', s''\ldots) = \frac{(s+s'+s''+\ldots)!}{s!\,s'!\,s''!\ldots}.$$

14. Illustrative Problems.

I. A bag contains $b$ black balls, and $b'$ white ones. A number $s$ of balls being drawn, what is the probability that they will all be white?

$\overset{s}{\underset{b+b'}{C}}$ will be the whole number of combinations of $s$ balls in the bag, any one of which may be drawn. Of these combinations, only $\overset{s}{\underset{b'}{C}}$ will consist entirely of white balls. The probability required is therefore $\overset{s}{\underset{b'}{C}} \div \overset{s}{\underset{b+b'}{C}}$. In the same way the probability of drawing $s$ black balls is found to be $\overset{s}{\underset{b}{C}} \div \overset{s}{\underset{b+b'}{C}}$.

Example. If there are two white balls and eight black ones, of which two are drawn, there will be forty-five combinations of 2 in 10, any one of which may be drawn. But only one of these combinations will consist entirely of white balls, and the probability of drawing that combination will be $\frac{1}{45}$, and the odds against it 44 to 1. We might arrive at the same result by the principle of § 10. If we suppose the balls to be drawn in succession, the probability that the first ball drawn will prove white is $\frac{1}{5}$, since of 10 balls 2 are white. On the supposition that this event occurs, the probability that the second ball will also be white is $\frac{1}{9}$, since of 9 balls one will

be white. The probability of the compound event is therefore $\frac{1}{9} \times \frac{1}{5} = \frac{1}{45}$.

II. To find the probability, that, at whist, a player other than the dealer holds any given number of trumps.

The trump turned up belongs to the dealer, there will remain 51 cards, 12 trumps, and 39 non-trumps to be divided at random, the dealer taking 12, and each of the players 13. In order that a player may have no trump, his 13 cards must come entirely from the 39 non-trumps, and the number of ways in which this result can happen is $\overset{13}{\underset{39}{C}}$. A hand containing one trump only may be composed of any one of the 12 trumps combined with any combination of 12 in the 39 non-trumps, and such a hand can happen in $12 \times \overset{12}{\underset{39}{C}}$ different ways. In general, a hand containing $s$ trumps may consist of any $s$ of the 12 trumps combined with any $13 - s$ of the 39 non-trumps, and there will be $\overset{s}{\underset{12}{C}} \times \overset{13-s}{\underset{39}{C}}$ such hands. The probability of holding $s$ trumps will therefore be $\overset{s}{\underset{12}{C}} \times \overset{13-s}{\underset{39}{C}} \div \overset{13}{\underset{51}{C}}$.

If, in the above expression, we give to $s$ the successive values 0, 1, 2, &c., we shall find

| | | | | | | |
|---|---|---|---|---|---|---|
| a probability | of | .02 | that the | player has | no | trump, |
| " | " | .10 | " | " | 1 | " |
| " | " | .23 | " | " | 2 | trumps, |
| " | " | .30 | " | " | 3 | " |
| " | " | .22 | " | " | 4 | " |
| " | " | .10 | " | " | 5 | " |
| " | " | .03 | " | " | 6 | " |
| a very small probability | | | | " | 7 | or more. |

By a process of reasoning similar to the above we shall find the

probability that the dealer has $s$ trumps to be

$$\overset{s-1}{\underset{12}{C}} \times \overset{13-s}{\underset{39}{C}} \div \overset{12}{\underset{51}{C}}.$$

III. A set of dice being thrown from a box, to find the probability that any given system of numbers will be thrown.

Suppose for the sake of clearness, that the several dice are distinguished as the first, second, &c. Then any one of the six sides of the first die may be combined with any one of the six sides of the second, making $6^2$ on two dice. Any one of these combinations may be combined with any side of the third, making $6^3$ combinations on three dice; and by continuing the same process it is seen, that, in general, the number of combinations on $n$ dice is $6^n$. The given system of numbers may be thrown in as many ways as there are permutations of the numbers composing it; and if we represent the number of permutations by $P$, the probability required will be $\frac{P}{6^n}$. This will be easily understood by comparing the following examples:

1. Suppose that there are three dice of different colors, — say white, yellow, red, and let it be required to find the probability that the numbers 1, 2, 3, will be thrown. This system can be thrown on the three dice in the six following ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| White, | 1 | 1 | 2 | 2 | 3 | 3 |
| Yellow, | 2 | 3 | 1 | 3 | 1 | 2 |
| Red, | 3 | 2 | 3 | 1 | 2 | 1 |

The probability of each way being $\frac{1}{216}$, the probability required will be $\frac{1}{36}$.

2. If two aces and a deuce were required to be thrown, it could be done only in the following ways:

| | | | |
|---|---|---|---|
| White, | 1 | 1 | 2 |
| Yellow, | 1 | 2 | 1 |
| Red, | 2 | 1 | 1 |

The probability of its being thrown is therefore $\frac{1}{72}$.

3. The probability of throwing three aces is $\frac{1}{216}$, since they can be thrown in only one way.

The proposition, that an ace, deuce, and three are six times as likely to be thrown as three aces, might at first sight seem paradoxical. A comparison of the above will, however, make it quite clear.

## NOTE ON MAXIMA AND MINIMA.

BY LEWIS R. GIBBES,
Professor of Mathematics in the College of Charleston, South Carolina.

The method used in the Note on the cycloid may be applied to other problems involving consideration of maxima or minima, and with advantage in such branches as are studied without supposing in the student a knowledge of the Calculus, or in questions which may be presented to him before he has yet entered upon the higher parts of his intended mathematical course.

We will apply it to an old problem, often mentioned in works on descriptive astronomy, but seldom solved, the result only being given, the student being supposed unacquainted with the Calculus. Observation shows that the planet Venus passing from inferior conjunction towards greatest elongation, increases in brilliancy, attains a maximum brightness at a certain point, and then declines as it advances towards superior conjunction. Hence the following

PROBLEM. *To find the position of a planet at the time of its greatest brilliancy, as seen from another whose orbit includes its own.*

We will suppose the orbits of both to be circular and in the same plane, which supposition will be found not to affect the generality of the conclusion.

Let $r$ be the radius of the orbit of the interior planet, and $r'$ the radius of the exterior; also, let $z$ be the distance between the two planets at any time, and $B$ the brightness at the time, the unit of brilliancy being the brightness of the unit of area of the disc of the planet, at the unit of distance from the Sun and from the exterior planet. By trigonometry, the cosine of the angle at the exterior planet is

(1) $$\cos \frac{r'}{z} = \frac{z^2 + r'^2 - r^2}{2\,z\,r'}.$$

The cosine of angle at interior planet is

(2) $$\cos \left(180° - \frac{r}{z}\right) = \frac{z^2 + r^2 - r'^2}{2\,z\,r}.$$

From optics, we have

$$B = \frac{\text{illuminated area of disc}}{z^2\,r^2} = \frac{\text{a semicircle} \pm \text{a semi-ellipse}}{z^2\,r^2},$$

the semi-major axis of the ellipse is equal to the radius of the semi-circle, and the semi-minor axis is equal to that radius multiplied by $\cos\left(180° - \frac{r}{z}\right)$, and when the semi-ellipse is subtractive, the cosine is negative, so that the illuminated area is always proportional to $1 + \cos\left(180° - \frac{r}{z}\right)$, and we may put $B = \frac{1 + \cos\left(180° - \frac{r}{z}\right)}{z^2\,r^2}$ or by (2) $= \frac{z^2 + 2\,z\,r + r^2 - r'^2}{2\,z^3\,r^3}$. Hence we have

(3) $$2\,r^3 B = \frac{1}{z} + 2\,r\,\frac{1}{z^2} + (r^2 - r'^2)\,\frac{1}{z^3}.$$

Let $z_0$ be the distance of the interior planet from the exterior at the point of maximum brilliancy; then there can be found pairs of points, one member of each pair on each side of that point, at which the brilliancy, though less than the maximum, must be equal. Let $z_1$ and $z_2$ be the distances of the interior planet from the exterior one at the two points of one of these pairs, and let $B_1$ be the brightness at those points, the same for each. Then we shall have these two equations similar in form to (3):

For the first point, $2r^3 B_1 = \frac{1}{z_1} + 2r\frac{1}{z_1^2} + (r^2 - r'^2)\frac{1}{z_1^3}$.

For the second point, $2r^3 B_1 = \frac{1}{z_2} + 2r\frac{1}{z_2^2} + (r^2 - r'^2)\frac{1}{z_2^3}$.

By subtraction, $0 = \frac{1}{z_1} - \frac{1}{z_2} + 2r\left(\frac{1}{z_1^2} - \frac{1}{z_2^2}\right) + (r^2 - r'^2)\left(\frac{1}{z_1^3} - \frac{1}{z_2^3}\right)$.

Multiplying by $\frac{z_1^2 z_2^2}{z_2 - z_1}$, $0 = z_1 z_2 + 2r(z_1 + z_2) + (r^2 - r'^2)\frac{z_1^2 + z_1 z_2 + z_2^2}{z_1 z_2}$.

As this equation holds good for every such pair of points, however near to the point of maximum brilliancy, it holds good when they coincide with it; but then $z_1 = z_2 = z_0$, and $z_1^2 = z_2^2 = z_1 z_2 = z_0^2$; hence, substituting in the above equation $z_0$ and $z_0^2$ for the quantities to which they are then equal, we have the equation of the maximum

$$0 = z_0^2 + 4rz_0 + 3(r^2 - r'^2).$$

Whence we get

(4) $z_0 = (r^2 + 3r'^2)^{\frac{1}{2}} - 2r$, or $\frac{z_0}{r'} = \left(\frac{r^2}{r'^2} + 3\right)^{\frac{1}{2}} - \frac{2r}{r'}$,

(5) $r = (\frac{1}{3}z_0^2 + r'^2)^{\frac{1}{2}} - \frac{2}{3}z_0$, or $\frac{r}{r'} = \left(\frac{1}{3}\frac{z_0^2}{r'^2} + 1\right)^{\frac{1}{2}} - \frac{2}{3}\frac{z_0}{r'}$,

rejecting the negative values as not conforming to the conditions of the problem. By substituting this value of $r$ for $r$ in (1), and of $z_0$ for $z$ in (2), we obtain

(6) $$\cos\frac{r'}{z_0} = \frac{2}{3}\left(\frac{1}{3}\cdot\frac{z_0^2}{r'^2} + 1\right)^{\frac{1}{2}} + \frac{2}{9}\cdot\frac{z_0}{r'},$$

$$\cos\left(180° - \frac{r}{z_0}\right) = \frac{3\frac{r}{r'} + \frac{r'}{r} - 2\left(\frac{r^2}{r'^2} + 3\right)^{\frac{1}{2}}}{\left(\frac{r^2}{r'^2} + 3\right)^{\frac{1}{2}} - 2\frac{r}{r'}}.$$

With any given value of $r$ less than $r'$, the distance at greatest brilliancy, $z_0$, can be obtained by (4), and the elongation at that time $\frac{r'}{z_0}$, and the annual parallax $\left(180° - \frac{r}{z_0}\right)$ from equations (6).

If $\left(180^\circ - \frac{r}{z_0}\right) = 90^\circ$, which can only happen at the greatest elongation, then $\cos\left(180^\circ - \frac{r}{z_0}\right) = 0$; hence from second of equations (6), $3\frac{r}{r'} + \frac{r'}{r} - 2\left(\frac{r^2}{r'^2} + 3\right)^{\frac{1}{2}} = 0$, and we get $\frac{r^2}{r'^2} = \frac{1}{5} = (0.447)^2$; $\frac{r}{r'}$ is also the sine of the greatest elongation in the orbit whose radius is $r$, so that when $\frac{r}{r'} = 0.447$ the maximum brilliancy occurs exactly at the greatest elongation, and the elongation is then $26^\circ.33'.53''$.

Since, by the conditions of the problem, $\frac{r}{r'}$ is always less than unity, the denominator of the right hand side of the last of equations (6) is always positive, and the sine of that side will depend on that of the numerator; this will be negative when $\frac{r^2}{r'^2} > \frac{1}{5}$ or $\frac{r}{r'} > 0.447$, and positive in the contrary case, and the negative cosine gives $\left(180^\circ - \frac{r}{z_0}\right) > 90^\circ$, or maximum brilliancy occurs between greatest elongation and inferior conjunction; the positive cosine shows that it occurs between greatest elongation and superior conjunction.

If $\frac{r}{r'} = \frac{1}{4}$, then $\cos\left(180^\circ - \frac{r}{z_0}\right) = +1$; or $\left(180^\circ - \frac{r}{z_0}\right) = 0^\circ$; that is, if $\frac{r}{r'} = \frac{1}{4}$ or less, the maximum brilliancy occurs only at superior conjunction.

If $z_0 = 0$, then from first of equations (6), $\cos\frac{r}{z_0} = \frac{2}{3}$; that is, however near the interior planet may approach the exterior, the elongation at which the greatest brilliancy will occur, cannot exceed $48^\circ.11'.22''$. Hence if $\frac{r}{r'}$ be

| | | | | |
|---|---|---|---|---|
| $< 1$ and $> 0.447$ | max. | between inf. conj. and gr. elong. | elong. | betw. $48^\circ.11'$ and $26^\circ.34'$ |
| $= 0.447$ | brill. | at greatest elongation, | will | $27^\circ.34'$ |
| $< 0.447$ and $> 0.250$ | will | between gr. elong. and sup. conj. | then | between $26^\circ.34'$ and $0^\circ$ |
| $=$ or $> 0.250$ | occur | at superior conjunction, | be | $0^\circ$ |

From this table, it will be seen that the statement made in some works (see Olmsted's Astronomy, article 311 of successive editions) is erroneous, that "an inferior planet is brightest at a certain point between its greatest elongation and inferior conjunction." The mean radius vector of Mercury being 0.387, he in general arrives at greatest brightness between greatest elongation and superior conjunction.

If the exterior planet be supposed fixed at a certain point in its orbit, the points, at each of which occurs the maximum brilliancy for successive values of $r$, will lie in a curve whose equation we proceed to find. Let the fixed position of the exterior planet be the origin of rectangular coördinates, the corresponding radius vector being the axis of $x$ positive towards the sun; and let $x$ and $y$ be the coördinates of any point in the curve whose distance from the superior planet is $z$, and from the sun is $r$, continuing to express by $r'$ the distance of the exterior planet from the sun. Then we have by

$$\left.\begin{array}{lll} \text{geometry} & & z^2 = y^2 + x^2 \\ (7) & \text{also} & r^2 = y^2 + (r' - x)^2 \\ \text{and from equation (4),} & & z^2 = 5r^2 - 4(r^4 + 3r'^2 r^2)^{\frac{1}{2}} + 3r'^2 \end{array}\right\}$$

and by combining these equations so as to eliminate $z$ and $r$, we will obtain the equation of the curve. This elimination will most easily be effected thus: Subtract the first equation from the sum of the second and third, simplify, clear of radicals and reduce, and there will result $(r' + x)r^2 = r'^2(r' - x) + \frac{1}{4}r'x^2$; now, by substituting the value of $r^2$ from the second of the above equations (7) and reducing, we get, finally, $y = \pm x\left(\frac{\frac{5}{4}r' - x}{r' + x}\right)^{\frac{1}{2}}$. This equation shows the curve to be a defective hyperbola of Newton's 41st species of lines of the third order.

Since to every value of $x$ there are two equal values of $y$ with contrary sines, the axis of abscissas is the axis of the curve, and,

since there is but one constant $r'$ in the equation, all such curves are similar. If $x$ be positive and greater than $\frac{5}{4}r'$, $y$ is impossible, no part of the curve being at a greater distance beyond the Sun from the exterior planet than $\frac{1}{4}$ of the radius of its orbit; if $x$ be positive and equal to $\frac{5}{4}r'$, then $y=0$, the curve cuts the axis of $x$ at a distance beyond the Sun equal to $\frac{1}{4}r'$, which corresponds to the case already mentioned of the maximum brilliancy at superior conjunction; the value of $y$ for $x=+r'$ is equal to that for $x=+\frac{1}{2}r'$, each equal to $r' \times \frac{1}{4}\sqrt{2}$, and this is also the value of $y$ corresponding to $x=-\frac{1}{4}r'$; the three values of $y$ for the three values of $x$, $+\frac{7}{8}r'$, $+\frac{1}{4}r'$ $-\frac{5}{8}r'$ are to each other in the ratio of 7, 4, 25; the value of $y$ for $x=-\frac{3}{4}r'$ is to that for $x=-\frac{1}{4}r'$ as 6 to 1; if $x=\frac{1}{8}r'$, then $y=x$; if $x=0$, then $y=0$; the origin of coördinates is a multiple point, two branches of the curve passing through it; lastly, if $x=-r'$, then $y$ is infinite, the Hyperbola has an asymptote, cutting the axis at right angles at a point distant from the Sun twice the radius of the orbit of the exterior planet. The curve comes in from infinity on one side of the axis, along the asymptote, bending gradually towards the point at which the exterior planet is supposed fixed, passes to the other side of the axis through that point, curves in a loop round the Sun, cutting the axis beyond the Sun at a distance already indicated, completes the loops by passing a second time through the fixed position of the exterior planet, and then, bending away from the axis, recedes to infinity, approaching the asymptote.

If the polar equation to the curve be desired, it can easily be had from the first of equations (6), the pole being at the same point as the origin of rectangular coördinates, $\frac{z_0}{r'}$ being the polar angle, reckoned from the axis of abscissas, and $\frac{z_0}{r'}$ being the radius vector, taking $r'$ as unity; we thus get $\frac{z_0}{r'}=\frac{9}{4}\cos\frac{z_0}{r'}-\sec\frac{z_0}{r'}$.

The portion of the curve belonging to the negative values of $x$ fulfil the algebraical and also geometrical conditions of the problem, but not the optical conditions; presenting another example of a class of cases, so puzzling to some persons, in which an algebraical expression, full of significance when applied to one question, is totally devoid of meaning when applied to another in which the fundamental conceptions are quite different. A planet in a circular orbit, seen from an interior one, will be brightest when in opposition.

So far the interior planet has been supposed to move in a circular orbit; if it move in an elliptic or other orbit, it will evidently be brightest at that point in its orbit which cuts the curve of maximum brilliancy belonging to the corresponding position of the superior planet, at the same moment. If there should be two or more such intersections at different points, there will be two or more maxima of brightness. The intensity of brightness will not be the same for both maxima; for if a planet were to move in the curve of maximum brilliancy from superior to inferior conjunction, the intensity of brightness at successive points, though a maximum for the given distance from the Sun at any one point, would diminish from superior conjunction to a point about equally distant from the Sun and from the exterior planet, at which would occur a *minimum maximorum*, and then increase again to inferior conjunction. At the *minimum maximorum*, where $z_0$ and $r$ would each be about $\frac{3}{5}$ of $r'$, the brightness would be little less than one fourth its intensity at the points where $r = \frac{1}{4} r'$ and $r = \frac{9}{10} r'$, at which two points the brilliancy would be nearly equal.

If the method above used be applied to determine the value of $x$ for which $y$ is a maximum, we shall find $x = + r' \times 0.763$, and the corresponding value of $y = r' \times 0.401$. An ellipse described about the Sun as a focus, having the radius vector to the fixed position of the

exterior planet as a line of apsides, the aphelion between the Sun and the superior planet, a perihelion distance equal to $0.250\,r'$, and a semi-minor axis equal to $0.401\,r'$, will closely approximate to the curve of maximum brightness throughout its perihelion half; so that a planet would always be near this curve through half its orbit, if that orbit were the above ellipse, whose mean distance would be $0.445\,r'$, and eccentricity $0.195\,r'$. The aphelion half would lie within the loop of the curve. The ellipse which would most closely approximate to the loop throughout its greatest extent would have its perihelion at the same point as that of the preceding one, a major axis of about $1.150\,r'$, and a minor axis of about $0.800\,r'$.

The above problem is as old as the time of Halley, who first proposed and solved it for the planet Venus, and his result is that generally quoted in the elementary works; he also remarked the limit which occurs when $r = \frac{1}{4}\,r'$. The other results given above we have not met with anywhere; but it is not impossible that, in the *Berlin Memoirs* or other equally inaccessible treasuries of science, all our results have been anticipated years ago, if not a century since. Even if so antiquated, they may be new to many of our readers.

## ARCS OF GREAT AND SMALL CIRCLES.

BY GEORGE P. BOND,
Director of the Observatory of Harvard College.

If a plane intersects the surface of a sphere without passing through its centre, a small circle, $c$, is formed by the intersection. The great circle, $C$, most nearly representing it through its entire circumference, will have its plane parallel to that of $c$. If it is only required to represent in the best manner an arc of $c$ less than a

circumference, by means of a corresponding arc of $C$, the planes of the two circles must be inclined by an angle depending on the length of the arc, and the distance of the plane of $c$ from the centre of the sphere. The direction of the line of intersection of the planes will be perpendicular to a diameter passing through the middle point of the arc.

In the case where $c$ differs but little from a great circle, let $\psi$ be the angle between the planes of $C$ and $c$; if $G$ be the great circle to which $c$ is secondary, $\psi$ will be the angle between $C$ and $G$. Let also $\zeta$ be the angle measured from their intersection to a point on $C$. The distance of $c$ from $C$ at this point measured on the surface of the sphere will be

$$(1) \qquad x = \psi \sin \zeta - \gamma,$$

$\gamma$ being the distance of $c$ from $G$.

In order to adjust an arc of $C$ comprised between the limits $\zeta_1$ and $\zeta_0$, so that the mean of the squares of the deviations of each infinitesimal element from its corresponding element of an arc of $c$ shall be the least possible; the position at $C$ must satisfy the conditions

$$(2) \qquad \psi = 4 \frac{\cos \zeta_0 - \cos \zeta_1}{(2\,\zeta_1 - \sin 2\,\zeta_1) - (2\,\zeta_0 - \sin 2\,\zeta_0)}\,\gamma. \qquad \frac{\zeta_1 + \zeta_0}{2} = 90°.$$

The general expression for the mean value of $x^2$ is

$$(3) \qquad \frac{\psi^2}{2} - \frac{\psi^2}{4}\left(\frac{\sin 2\,\zeta_1 - \sin 2\,\zeta_0}{\zeta_1 - \zeta_0}\right) + 2\,\psi\,\gamma\,\frac{\cos \zeta_1 - \cos \zeta_0}{\zeta_1 - \zeta_0} + \gamma^2,$$

which becomes, when the adjustment is made in conformity with the method of least squares

$$(4) \qquad \gamma^2\left(1 - \frac{2\,\eta^2}{1 + \eta \cos \theta}\right).$$

where

$$\theta = \frac{\zeta_1 - \zeta_0}{2}, \quad \eta = \frac{\sin \theta}{\theta}.$$

If we take the square root of the mean of the square of the

deviations of each of its elements for the mean error, irrespective of signs, of the arc of $C$ considered as representing a corresponding arc of $c$, we have from (4) the following results:

$$\begin{array}{llllll} & \text{Included arc, } \zeta_1 - \zeta_0 = & 0^\circ & \text{Mean error} = & \pm & 0.0000\,\gamma, \\ & \text{“} \qquad \text{“} & = 30 & \text{“} \qquad = & & 0.0100\,\gamma, \\ (5) & \text{“} \qquad \text{“} & = 60 & \text{“} \qquad = & & 0.0421\,\gamma, \\ & \text{“} \qquad \text{“} & = 90 & \text{“} \qquad = & \pm & 0.0975\,\gamma. \end{array}$$

It is evident that where the included arc as well as $\gamma$ is small, the approximation reaches a high degree of accuracy, the error varying nearly as the product of $\gamma$ into the square of the arc.

The above conclusions may be applied to the theory of the transit instrument. The path described by the optical axis of the transit is the small circle, $c$, distant from the great circle, $G$, to which it is secondary by the small arc $\gamma$. $G$ is commonly referred to the meridian by the small arcs $\alpha$ and $\beta$, which are its distances from the meridian at the horizon and at the zenith respectively. If $\varphi$ is the latitude of the place of observation, and $\delta$ the declination of the object towards which the instrument is directed, the deviation of the small circle $c$ from the meridian will be

$$(6) \qquad \alpha \sin(\varphi - \delta) + \beta \cos(\varphi - \delta) + \gamma\,;$$

which may be reduced to the form

$$(7) \qquad P \sin(Q - \delta)\,,$$

$P$ and $Q$ being constant for arcs for which the errors of deviation of $c$ from $C$ are insensible. With the usual adjustment of the instrument we may infer from (5) that its path for arcs of thirty or forty degrees may be treated as a great circle without appreciable error.

The same general conclusion may be reached in another way. If $\alpha$ and $\beta$ are eliminated from the equations

$$\begin{aligned}
&\alpha \sin(\varphi - \delta_1) + \beta \cos(\varphi - \delta_1) + \gamma = \mu_1, \\
(8) \qquad &\alpha \sin(\varphi - \delta_2) + \beta \cos(\varphi - \delta_2) + \gamma = \mu_2, \\
&\alpha \sin(\varphi - \delta_3) + \beta \cos(\varphi - \delta_3) + \gamma = \mu_3,
\end{aligned}$$

we have

$$(9) \qquad 4 \sin \frac{\delta_1 - \delta_2}{2} \sin \frac{\delta_1 - \delta_3}{2} \sin \frac{\delta_2 - \delta_3}{2} \gamma = \mu'''.$$

The coefficient of $\gamma$ is a small quantity of the third order when the included arcs are of the first order, while the second member of (9) will be affected by errors of observation multiplied by small coefficients of the first order. When $\gamma$ is small, of the order of errors of observation, it is plain that for arcs of moderate extent, both (8) and (9) may be satisfied by the value

$$\gamma = 0,$$

which reduces the path described, within the limits assigned, to an arc of a great circle.

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## ON MR. COLLINS'S PROPERTY OF CIRCULATES.*

BY JAMES EDWARD OLIVER, Lynn, Mass.

LET $C\xi$ denote a circulate of $\xi$ places; that is, a number formed, except perhaps its left-hand portion, by the endless repetition of $\xi$ figures in the same order. Let $F(\;)$ and $M(\;)$ be the greatest common factor and the least common multiple of the inclosed numbers. Let $l_1, l_2$, &c. be subfactors or multiples of certain of the $A$ numbers $x, y..z$, or of previous values of $l_s$; and form $\lambda_1, \lambda_2$, &c. in a precisely corresponding manner from $\xi, \eta,.\zeta$. Let the arbitrary integers, $w, w_1$, &c. be exact divisors of any numerators written over them; — and call $a$ the base of the system of numeration employed.

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* See page 295.

*From the periods of several circulates, to find the period of their product.* The $A$ conditions

(1) $$\frac{x'}{x} = C\xi,\quad \frac{y'}{y} = C\eta,\quad ..\frac{z'}{z} = C\zeta,$$

the fractions and the periods being in their simplest terms, are equivalent to

(2) $$a^{\xi} \equiv 1\,[x],^{*}\quad a^{\eta} \equiv 1\,[y],\quad ..a^{\zeta} \equiv 1\,[z];$$

where, as in all that follows, the exponents are assumed to be *minimum* positive roots of their congruences. Whence

(3) $$a^{F(\xi,\eta,.\zeta)} \equiv 1\,[F(x,y,.z)],$$

because all the roots of $a^{\varrho} \equiv 1\ [F(x,y,.z)]$ (3′), such as $\xi,\eta,.\zeta$, are multipliers of $\varrho_0$, the least positive root; hence their greatest common measure $F(\xi,\zeta,.\eta)$ is a multiple of $\varrho_0$; hence $F(\xi,\eta,.\zeta)$ is a root of (3′).

Again, since modulus $x$ requires that the exponent of $a$ be a multiple of $\xi$, while modulus $y$ requires the exponent to contain $\eta$, and so on, we have

(4) $$a^{M(\xi,\eta,.\zeta)} \equiv 1\ [x,y,.z,\ \therefore\ M(x,y,.z)].$$

From (3) and (4),

(5) $$a^{\lambda_s} \equiv 1\,[l_s].\qquad (6)\ \therefore\ l_s = \frac{a^{\lambda_s}-1}{w_s}.$$

From (4), $$\left(a^{M(\xi,,\zeta)} = 1 + w'.M(x,,z)\right)^{\frac{x..z}{M(x,,z)}}$$

$$= 1 + w'.x..z\left(1 + \frac{(x..z - M(x,,z))}{1.2}w' + \text{\&c.}\right).$$

But $\dfrac{(x..z - M(x,,z))..(x..z - s.M(z,,z))}{1.2\quad ...\quad (s-1)}$ is an integer; for since

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* See note at end of this article.

$M(x,,z)$ divides $(x..z)$, one of every $k$ consecutive terms in the numerator is divisible by $k$. Hence

(7) $$a^{M(\xi,,\zeta)\frac{x..z}{M(x,,z)}} \equiv 1 + w''.x..z \qquad \therefore \equiv 1\ [x..z].$$

(8) $$\therefore C\xi.C\eta..C\zeta = \frac{x'..z'}{x..z} = C.\frac{x..z}{M(x,,z)}M(\xi,,\zeta).$$

It remains to select such forms of $l_1$, $l_2$,, that $\frac{x..z}{M(x,,z)}$ may be some simple function $\psi$ of them; then by substituting for $l_1$, $l_2$,, their values (6), the period of $C\xi...C\zeta$ is expressed in terms of $\lambda_1, \lambda_2$,,; that is, of $\xi,,\zeta$.

Consider the $2^A-1$ factors, $2^{A-1}$ of which enter each letter $x,,z$; one entering only $y$, &c., and one entering all $x,,z$. Usually most of these factors are unity. Call $P_s$ the product of the $\frac{A..(A-s+1)}{1.....s}$ factors that enter $s$ letters each. $M(x,,z) = P_1P_2..P_A$, $x..z = P_1P_2^2..P_A^A$, $\therefore \psi = P_2P_3^2..P_A^{A-1}$, and requires at least $A-1$ functions $l_s$, even if any $l_s$ be squared, cubed, &c., since no $l_s$ is above the first degree in $P_A$ or $P_s$.

We may take $l_s$ = the least common multiple of the $\frac{A..(A-s+1)}{1..s}$ greatest common factors of $x,,z$ taken $s$ at a time, $\therefore l_s = P_sP_{s+1}..P_A$. Or $l_s$ may variously be taken unsymmetrical as to $(x_1,,x_A = x,,z)$; for instance $l_2 = F(x_2,x_1)$, $l_3 = F(x_3, M\overline{x_2,x_1})$, $l_4 = F(x_4, M\overline{x_3,,x_1})$ &c.; or &c.; all which may be further varied by permutating any of the letters $x,,z$ that enter unsymmetrically. In these cases, $\psi = l_2..l_A$, and (8) becomes

(9) $$C\xi..C\zeta = C.\frac{(a^{\lambda_2}-1)..(a^{\lambda_A}-1)}{w}M(\xi,,\zeta),$$

where $w = w_2..w_A$, and whose constants $\lambda_2,,\lambda_A$ are found from $\xi,,\zeta$ as $l_2,,l_A$ would be from $x,,z$. In the symmetrical case, $\lambda_s$ is the product of those highest powers of the primes $\alpha, \beta, \gamma$, which occurs

each in some $s$ of $\xi,, \zeta$; so that each $\lambda$, $\therefore$ also each $a\lambda - 1$, divides all preceding ones.

EXAMPLE. If $\varkappa_1,, \varkappa_A$ be all prime to each other,

$$(10) \qquad C\varkappa_1\varrho \,.. \, C\varkappa_A\varrho = C.\frac{(a^\rho - 1)^{A-1}}{w}\varkappa_1 \,..\, \varkappa_A\varrho.$$

$\lambda_s$ may exceed $A - 1$ in number, or the form of $\psi$ may vary, or both; as,

$$(11)\ C\xi.C\eta.C\zeta = C\left(\frac{w'}{Q^{P_3} - 1}\cdot\frac{(a^{\lambda_\xi} - 1)(a^{\lambda_\eta} - 1)(a^{\lambda_\zeta} - 1)}{w}M(\xi,\eta,\zeta)\right),$$

where $\lambda_\xi = M(F\overline{\xi,\eta}, F\overline{\xi,\zeta}) = F(M\overline{\xi,\eta}, M\overline{\xi,\zeta})$, $\lambda_\eta$ and $\lambda_\zeta =$ &c.

$(9 - 11)$ &c. give all the *simple* periods of the required product, but may also give other, *merely multiple* periods, a criterion for whose suppression would be desirable. For (3) and (4), though *necessary*, are not *sufficient* to (2). Hence (9), (11), &c., or even the different *forms* of either that come from different values of $\lambda_s$ may not be identical; and any factor not common to all, should be suppressed by conditioning the divisors and multipliers $w_s$, $w^{(s)}$.

Mr. COLLINS's case of (10), that

$$C\varkappa\varrho\,.\,C\tau\varrho = C.\frac{a^\rho - 1}{w}\varkappa\tau\varrho,$$

$\varkappa$ being prime to $\tau$, gives by inversion the period of the quotient of two circulates. If $C\theta = C\pi \div C\chi$, divide $\chi$ into factors $\varkappa, \varrho$, let $\theta = \tau\varrho$, and have $\tau$ prime to $\varkappa$.

$$C\varkappa\varrho\,.\,C\tau\varrho = C\pi. \qquad \therefore \frac{a^\rho - 1}{w}\varkappa\tau\varrho = \pi.$$

$$(12) \qquad \therefore\ \theta = \frac{w}{a^\rho - 1}\frac{\tau\varrho}{\chi}, \quad \left(\frac{\theta}{\varrho} \text{ prime to } \frac{\chi}{\varrho}\right),$$

where $\varrho$ is successively all factors of $\chi$, but $w < a^\rho$ may need further restriction. By similar steps, if $C\theta = \dfrac{C\pi}{C\chi_1 \,..\, C\chi_A}$, we divide $\chi_1,, \chi_A$

and $\theta$ into factors $\varkappa_1 \varrho, \varkappa_A \varrho$ and $\tau \varrho$, having $\tau$ prime to $\varkappa_1,, \varkappa_A$. Let $\lambda_2 = F(\chi_2, \chi_1)$, $\lambda_3 = F(\chi_3, M(\chi_2, \chi_1))$, &c. Since $\lambda_{A+1} = F(\theta, M(\chi_A,; \chi_1))$ reduces to $\varrho$, and $M(\chi_1,, \chi_A, \theta)$ to $\frac{\theta}{\varrho} M(\chi_1,, \chi_A)$, — (9) finally gives

$$(13) \quad \theta = \frac{w}{(a^{\lambda_2}-1)..(a^{\lambda_A}-1)(a^{\rho}-1)} \cdot \frac{\pi \cdot \varrho}{M(\chi_1,, \chi_A)} \left( \frac{\theta}{\varrho} \text{ prime to } \frac{\chi_1}{\varrho},, \frac{\chi_A}{\varrho} \right).$$

NOTE. — GAUSS's notation, $x \equiv y$ (mod. $z$), which we shall write $x \equiv y\,[z]$, is read "$x$ is congruous to $y$ with regard to the modulus $z$," or "$x$ leaves the same remainder as $y$ does, when divided by $z$." The modulus may be omitted when there is no fear of mistaking it. $x \equiv y\,[z', z'', z''', \therefore u]$ may be read, "$x$ leaves the same remainder as $y$, whether divided by $z'$, by $z''$, or by $z'''$; hence also when divided by $u$."

The second member of $a^{\zeta} \equiv y$ runs through a regular cycle as $\zeta$ increases; so that $a^{\zeta} \equiv b$ has an infinity of roots of the form $\zeta = \zeta_0 + w \cdot \zeta'$, $\zeta_0$ being the least positive root, and $\zeta'$ that of $a^{\zeta} \equiv 1$.

If $\frac{p}{q}$ be a circulate of $\chi$ places, the division of $p$ (or $\therefore$ of 1.000 &c.) by $q$ gives by definition a remainder of $p$ (or $\therefore$ of 1) after every $\chi$ quotient figures; hence, $a^{\chi} \equiv 1\,[q]$.

See GAUSS's *Disquitiones Arithmeticæ*, or SERRET's excellent *Algebre Superieure*. (Paris, 1843), &c.

## SOLUTIONS OF PROBLEMS IN PROBABILITIES.

By SIMON NEWCOMB, Nautical Almanac Office, Cambridge.

1. "*A* HAS the reputation of telling the truth as often as three times in four, *B* as often as four times in five, and *C* as often as four times in seven. When *A* and *B* agree in affirming what *C* denies, what is the probability that *A* and *B* tell the truth?" See p. 235.

SOLUTION. The *a priori* probability that *A* and *B* tell the truth and *C* falsifies is $\frac{3}{4} \times \frac{4}{5} \times \frac{2}{7} = \frac{6}{35}$. The *a priori* probability that *A* and *B* falsify and *C* tells the truth is $\frac{1}{4} \times \frac{1}{5} \times \frac{5}{7} = \frac{1}{28}$. The proba-

bility of the proposition asserted by $A$ and $B$ is therefore $\frac{6}{35} \div (\frac{6}{35} + \frac{1}{28}) = \frac{24}{29}$.

2. "A person goes on throwing a common dice until he throws an ace; at whatever throw this occurs (the $n$th), he is to receive the $n$th of a dollar. What is the value of his expectation?" See p. 235.

Solution. The probability that an ace will be thrown on any given trial is $\frac{1}{6}$, and the probability against it is $\frac{5}{6}$. The probability, then, that an ace will be thrown on the $n$th throw and not before is $\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \ldots . = \frac{1}{6}(\frac{5}{6})^{n-1}$; the probability being compounded of the one favorable and the $n-1$ unfavorable cases. Since the thrower gets $\frac{1}{n}$th of a dollar on the occurrence of this event, the value of his expectation on the $n$th throw is $\frac{1}{n} \times \frac{1}{6}(\frac{5}{6})^{n-1}$. Giving $n$ successively all integer values from 1 to $\infty$, the complete value of his expectation is

$$\tfrac{1}{6}(1 + \tfrac{1}{2} \cdot \tfrac{5}{6} + \tfrac{1}{3} \cdot (\tfrac{5}{6})^2 + \tfrac{1}{4}(\tfrac{5}{6})^3 + \tfrac{1}{5}(\tfrac{5}{6})^4 + \&c.) = \tfrac{1}{5} \text{ nep. log } 6.$$

The value of his expectation is therefore 63 cents, nearly.

## Mathematical Monthly Notices.

*Astronomical Notices.* Nos. 1–6. Edited by Dr. F. Brünnow, Ann Arbor, Michigan.

This Periodical is issued in numbers of eight pages each, 8vo, and twenty-four numbers will form one volume; but, when desirable, it will be issued more frequently in semi-numbers, four pages at a time. The Editor says: "The main object I have in view in publishing this new periodical is to secure the regular publication of observations made at the Observatory at Ann Arbor, and of the scientific investigations of the officers of the Observatory in general. It is also my intention to make these Notices as useful as possible to practical astronomers, by furnishing them always in the shortest time, as well with reliable ephemerides of newly discovered comets and asteroids for the whole time during which they remain visible, as with proper comparison stars observed here in advance, whenever it is possible."

There could be no doubt of the value of such a periodical in the hands of Dr. Brunnow, even if the evidence were not before us in the six numbers already issued; and it cannot fail to exert

a beneficial influence upon the students of the University. For every new discovery, every reliable observation, every new orbit, in fact, for every result worthy of permanent record, they have at once the means of publication; and it seems to us that this periodical must act as an additional incentive to draw students of astronomy to this flourishing University.

The objections, as it seems to us, which can be urged against its establishment, are the inconvenience of being under the necessity of consulting so many different journals, and the fact, that it is with difficulty that one devoted especially to astronomy can be sustained in this country.

*Biographical and Literary Dictionary of the History of the Exact Sciences;* containing References to the Relations and Developments in Mathematics, Astronomy, Physics, Chemistry, Mineralogy, Geology, &c., in all Times and Countries; compiled by J. D. POGGENDORFF, Fellow of the Academy of Sciences of Berlin.

This very important work is to be published in four parts, making a volume of from 1,000 to 1,200 large double-column pages. The first and second parts, of 288 pages each, containing about four thousand names, exclusive of references, and ending with the article "HUDDART," we have examined with sufficient care to be able heartily to recommend it to all in this country devoted to science, or interested in its progress. The spirit of the work may be most clearly and briefly intimated in the language of the compiler.

"The leading principle for this dictionary has been to include all persons connected with the mathematical and inductive sciences, as far as any certain notices of their lives could be gained: a condition securing to the work its biographical-literary character, without permitting it to degenerate into a mere list of names and books. Moreover, it was not the intention of the author to give strict biographies and complete literary indices; such would have inconveniently increased the bulk of the work beyond the power of a single compiler, and disadvantageously restricted its circulation and usefulness; he purposed rather to present, in short sketches, a summary, such as has not yet appeared,— a manual, which may be in the possession of every friend of the inductive sciences, satisfying him on the chief points of date, life, and works of persons active in the field, and providing him at the same time with references to the sources whence more detailed information may be obtained.

"For the last ten years the author has been continually employed in the compilation of this work, in which task he has been particularly aided by the extensive literary-historical treasures of the royal library of Berlin, as well as by the services of several friends who have kindly supplied him with numerous authentic communications from scientific men of the present day. Taking into view the copious material thus collected, it may be confidently asserted that this work will be inferior to no similar one on any other branch of science; and representing as it does a whole library of biographical resources, it will doubtlessly animate to historical study in this sphere, and tend to render the same fruitful."

*Ueber die Verbesserung der Planeten-Elemente aus beobachteten Oppositionen, angewandt auf eine neue Bestimmung der Pallas-Bahn.* Von Dr. J. G. GALLE, ordentlichen Professor der Astronomie an der Universitat zu Bresleau.

In the Memoir before us, we have a new determination of the orbit of Pallas. The corrections of the osculating elements for 1810 are based upon twelve oppositions; those best observed between 1816 and 1855. The new orbit now agrees as well with observation as could be expected, considering the length of time elapsed, and the fact that the perturbations by Saturn and Mars have not been taken into account.

The author has used the method given by GAUSS (Theoria Motus, Art. 76) for forming the

equations of condition; and although it is not the shortest, it is one which the student can readily comprehend. A short historical sketch, which precedes the investigation, informs us that GAUSS was the first to subject the orbit of Pallas to a systematic investigation, as he did those of the three other asteroids discovered before 1810; that ENCKE next took it, and has now put it in charge of Dr. GALLE, whose investigations have already shown that it could not have fallen into abler hands.

It is to be hoped, that tables of this interesting planet may soon be prepared; which has already been done for Flora and Victoria, by Dr. BRUNNOW; for Egeria, by Prof. PEIRCE; for Astræa and Hygea, by Prof. ZECH; and for Metis and Lutetia, by Mr. OTTO LESSER.

Our readers will recall Dr. GALLE as the astronomer to whom LEVERRIER first communicated his predicted place of Neptune, by means of which the planet was at once identified.

## Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the April Number of the MONTHLY.

DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y., answered all the questions.

JOHN N. BENTON, Earleville, N. Y., answered all the questions.

WILLIAM C. HENCK, Student in Dedham High School, answered I. and II. (CARLOS SLAFTER, Principal.)

C. HERSCHEL, Student in the Lawrence Scientific School, answered all but V.

W. M. STIRLING, Baltimore, Md., answered all but III. and V.

E. W. NEWTON, Student in Marietta College, Ohio, answered all but III. and V. (E. W. EVANS, Professor.)

B. F. CLARKE, Student, Waltham, Mass., answered question I. (Rev. T. HILL, Teacher.)

WM. EGERTON, Student in Baltimore College, answered all the questions. (RICHARD COTTER, Professor.)

O. B. WHEELER, Student in the University of Michigan, Ann Arbor, answered all the questions. (D. WOOD, Professor.)

CHARLES BETTLE, Sophomore Class, Haverford College, West Haverford, Pa., answered all the questions but V. (M. C. STEVENS, Professor.)

CHARLES W. HASSLER, Columbian College, Washington, D. C., answered all the questions but III. and V. (EDWARD T. FRISTOE, Professor.)

HORACE OTIS, Adams Centre, N. Y., answered all the questions but V.

MARQUIS HALL, Brimfield, Mass., answered all the questions but III. and V.

ASHER B. EVANS, Madison University, Hamilton, N. Y., answered all the questions.

JAMES B. FOSSETT, Student in New London Institute, answered all the questions but III. and V. (EPHRAIM KNIGHT, Professor.)

L. E. NEWCOMB, East Machias, Me., answered all the questions but III. and V.

J. C. ELLIOTT, Junior Class, Indiana University, Bloomington, answered questions I. and IV. (DANIEL KIRKWOOD, Professor.)

J. W. JENKS, Senior, Columbia College, New York City, answered all the questions but III. and V. (WILLIAM G. PECK, Prof.)

THE

# MATHEMATICAL MONTHLY.

Vol. I....AUGUST, 1859....No. XI.

## PRIZE PROBLEMS FOR STUDENTS.

### I.

SOLVE the equations

$$x + y = a$$
$$(x^3 + y^3)(x^2 + y^2) = b,$$

and give a discussion of the values of the roots.

### II.

Let $A$, $B$, $C$ be the angles, and $a$, $b$, $c$ the opposite sides, of a plane triangle; it is required from the relation

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

to deduce the formula

$$a^2 = b^2 + c^2 - 2\,b\,c \cos A.$$

### III.

A number $n$ of equal circles touch each other externally, and include an area of $a$ square feet; to find the radii of the circles. (Communicated by ARTEMAS MARTIN, Esq.)

### IV.

If the sides of a spherical trapezium be denoted by $a$, $b$, $c$, $d$, the

diagonals by $\delta_1$ and $\delta_2$, and the distance between the middle points of the diagonal by $\Delta$; show that

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \tfrac{1}{2}\, \delta_1 \cos \tfrac{1}{2}\, \delta_2 \cos \Delta.$$

(Communicated by GEORGE EASTWOOD, Esq.)

V.

From an urn containing four white and four black balls, four are repeatedly drawn and replaced. *A* agrees to pay *B* one dollar every time the four balls drawn are equally divided between white and black; but if three, or all four, are of the same color, *B* is to pay *A* one dollar. Who has the advantage, and what is its value for each drawing? (Communicated by SIMON NEWCOMB, Esq.)

The solution of these problems must be received by the first of October, 1859.

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## REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VII., Vol. I.

THE first Prize is awarded to O. B. WHEELER, student in the University of Michigan, at Ann Arbor.

The second Prize is awarded to WILLIAM EGERTON, student in the Baltimore College, Baltimore, Maryland.

### PRIZE SOLUTION OF PROBLEM I.

"On two sides *AC* and *BC* of any triangle *ABC*, let any parallelograms *ACDE* and *BCFG* be described. Let *ED* and *FG* produced, meet in *H*; join *HC*, and through *A* and *B* draw *AL* and *BM* equal and parallel to *HC*. Join *LM*. It is required to prove that the parallelogram *ALMB* on the side *AB* is equal to the sum of the parallelograms on the sides *AC* and *BC*.

"Show also that the Pythagorean proposition is a particular case of this proposition."— PAPPUS.

The parallelograms $AH$ and $AD$ are equivalent; so are the parallelograms $BH$ and $BF$, since they have the same base and altitude. Therefore $LH = AC$, and $HM = CB$. But since $AL$ and $BM$ are each equal and parallel to $HC$, they are equal and parallel to each other, and $LM = AB$. Therefore the triangle $LHM$ is equal to $ABC$, since they are mutually equilateral. Take away the triangle $OCP$ from each, and we have left $LHMPCO$, equivalent to $AOPB$. Add $ALO$ and $BMP$ to both, and we have

Fig. 1.

$$ALHC + BMHC = AEDC + BGFC = ALMB.$$

Or thus: Produce $HC$ till it meets $AB$ in $I$; then will $AD = AH = AK$. For $AI = LK$, and the triangles $LHK = ACI$. Taking away the common part $OCK$, and adding $ALO$ to both remainders, we have $AH = AD = AK$. Also $BF = BK$. $\therefore AD + BF = AM$.

Corollary. The parallelograms $AK$ and $BK$ have the same altitude, and are therefore to each other as their bases $AI$ and $BI$. Hence $AD : BF :: AI : IB$; that is, the line $HC$ produced cuts the third side $AB$ into parts which are to each other as the parallelograms described on the adjacent sides.

Cor. 2. If triangles be described on the sides $AC$ and $BC$, and lines be drawn through their vertices $Q$ and $R$ parallel to the sides $AC$ and $BC$, and produced till they meet in $H$, and the remaining lines be drawn as in the figure, then the vertex of the triangle described on $AB$, which shall be equivalent to the sum of the other two, will be found anywhere in the line $LM$; for a triangle is equivalent to half the parallelogram having the same base and altitude.

Cor. 3. The triangles $AQC$ and $BRC$ are to each other as $AI:BI$, since the triangles are the halves of $AD$ and $BF$.

Cor. 4. The triangle $ASI$ is equivalent to $AQC$, and $BSI$ to $BRC$.

Cor. 5. When $ACB$ is right-angled at $C$, and squares are described on $AC$ and $BC$, the diagonal $HC$ equals $AB$. For the two triangles $CDH$ and $ACB$ have $AC=CD$, $BC=DH$, and $C$ and $D$ right angles. $\therefore CH=AB=AL$. Also the angle $DCH=CAB$, and $DHC=CBA=HCF=LAC$. $\therefore DCH+DHC=CAB+LAC$ $=$ a right angle. $ALMB$ is therefore a square equivalent to the sum of the other two.

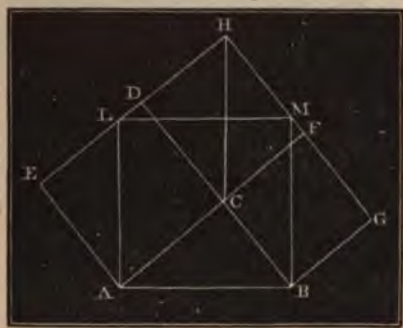


Fig. 2.

This solution is by David Trowbridge. In most of the other solutions, $HC$ (Fig. 1) was produced to $I$; then $AD=AH=AK$, and $BF=BH=BK$, as is easily seen. Therefore, by addition, $AD+BF=AM$.

### Prize Solution of Problem II.

"The area of a right-angled triangle is equivalent to the rectangle of the differences between the radius of the inscribed circle and the two shorter sides respectively; or the rectangle of the segments of the hypothenuse made by a perpendicular let fall upon it from the centre of the inscribed circle."


Let $b$ be the base and $a$ the altitude of the triangle $ABC$, and $r$ the radius of the inscribed circle. It will be readily seen that the triangle $GDA=ADE=DAH$. $\therefore GDEA=EDHA$. $GDC=CDF=DCI$. $\therefore GDFC=FDIC$.

$$ACB=CIDHAB=ar+(b-r)r$$
$$=ar+br-r^2.$$

$$KCA = ACB = ab - (ar + br - r^2) = ab - ar - br + r^2;$$

but $\quad KIDH = (a - r)(b - r) = ab - ar - br + r^2.$

$$\therefore ABC = KIDH = (a - r)(b - r) = AE \times CF = AG \times CG.$$

This solution is by C. HERSCHEL.

### PRIZE SOLUTION OF PROBLEM III.

"The distance between two points, $A$ and $B$, is $a$ miles. A person starts at $A$ and travels the first day one $m$th his distance to $B$; the second day he travels back one $m$th his distance to $A$; the third day he turns and travels one $m$th his distance to $B$, and so on. How far will he travel in $n$ days, and how far will he be from $A$?"

Let $a$, $a_1$, $a_2$, &c., denote his distances from the point $A$ or $B$ towards which he travels on the successive days; and let $x_1$, $x_2$, $x_3$, &c., denote the distances travelled. Then, by the question

$$(1) \qquad \frac{a}{m} = x_1, \ \frac{a_1}{m} = x_2, \ \frac{a_2}{m} = x_3, \text{ \&c.,}$$

and by addition

$$(2) \quad \frac{1}{m}(a + a_1 + a_2 + \ldots a_{n-1}) = x_1 + x_2 + x_3 + \ldots x_n = S_n$$

the whole distance travelled in $n$ days. Next we have

$$
\begin{array}{lll}
a = a, & m x_1 = m x_1, & x_1 = x_1 \\
a_1 = a - a + x_1, & m x_2 = m x_1 - m x + x_1, & x_2 = x_1 - \frac{m-1}{m} x_1, \\
(3) \ a_2 = a - a_1 + x_2, & m x_3 = m x_1 - m x_2 + x_2, & x_3 = x_1 - \frac{m-1}{m} x_2, \\
\vdots & \vdots & \vdots \\
a_n = a - a_{n-1} + x_n, & m x_{n+1} = m x_1 - m x_n + x_n, & x_{n+1} = x_1 - \frac{m-1}{m} x_n.
\end{array}
$$

By addition, the third set gives

$$S_n + x_{n+1} = (n + 1) x_1 - \frac{m-1}{m} S_n;$$

hence, and by (1)

$$(4) \qquad S_n = \frac{(n+1) m x_1 - m x_{n+1}}{2m - 1} = \frac{(n+1) a - a_n}{2m - 1}.$$

To find the value of $a_n = m x_{n+1}$, put $\frac{m-1}{m} = p$, and by succes-

sive substitutions, the last set of equations (3) give

$$(5)\quad \begin{aligned} &x_1 = x_1,\ x_2 = x_1(1-p),\ x_3 = x_1(1-p+p^2), \text{ and so on to} \\ &x_{n+1} = x_1(1-p+p^2-p^3+p^4-\ldots\ldots \pm p^n). \end{aligned}$$

The upper sign is to be used when $n$ is *even*, and the lower when it is *odd*. Summing the series (5) becomes

$$(6)\quad x_{n+1} = x_1 \frac{1 \pm p^{n+1}}{1+p}, \text{ and } m\,x_{n+1} = a_n = a\frac{1 \pm p^{n+1}}{1+p};$$

or, replacing the value of $p$,

$$(7)\qquad a_n = m\,x_{1+n} = a\frac{m^{n+1} \pm (m-1)^{n+1}}{m^n(2m-1)}.$$

Substituting this value of $a_n$ in (4) and reducing, we get

$$(8)\qquad S_n = \frac{(n+1)\,a}{2m-1} - \frac{m^{1+n} \pm (m-1)^{n+1}}{m^n(2m-1)^2}.$$

When $n$ is odd, (7) denotes the distance from $A$; but when $n$ is even, this distance is $a - a_n$. This solution is by DAVID TROWBRIDGE.

### PRIZE SOLUTION OF PROBLEM IV.

"The volume of any right cone equals the product of its whole surface by one third the radius of the inscribed sphere." — Communicated by Prof. SNELL.

Let a sphere be inscribed in the right cone. Circumscribe a regular polygon about the base of the cone, and join the vertices of the polygon with the vertex of the cone. The faces of this circumscribed pyramid will be tangents to both the cone and sphere. Join the vertices of the solid angles of the pyramid and the centre of the sphere, and thus divide it into as many other pyramids as it has faces. All these pyramids have a common vertex at the centre of the sphere, and their common altitude is the radius of the sphere, since all their bases are tangents to it. The solidity of each pyramid is the product of its base by one third of the radius of the sphere, and therefore the solidity of the pyramid circumscribing the cone is the product of its whole surface by one third of the radius of the sphere. Let, now,

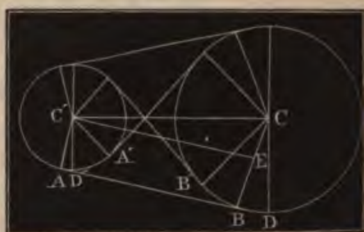
the number of sides of the polygon circumscribing the cone's base be continually increased; the limit of the polygon is the base of the cone; and the limit of the whole surface of the pyramid is the whole surface of the cone. Hence the volume of the cone is the product of its whole surface by one third of the radius of the inscribed sphere.

This solution is by J. C. Elliott; and the same reasoning was used by O. B. Wheeler.

### Prize Solution of Problem V.

"One pulley drives another by means of a belt; give the length of the belt $l$, the diameter $D$, of the larger pulley, the distance $a$ between the centres of the pulleys; to find the diameter $d$ of the smaller pulley. Find also a simple approximate formula for the use of machinists."

Let $CB = R = \frac{1}{2} D$, and $C' A = r = \frac{1}{2} d$. If $b, D$, and $a$ remain the same, it is evident that $d$ will have different values according as the belt is a crossed or open one. For a crossed belt the sum of the straight parts is $2\sqrt{a^2 - (R+r)^2}$, since each is a leg of a right-triangle, having $a$ for the hypothenuse and $R + r$ for the other leg. In this triangle the angle opposite $R + r$ equals $B' C D$ or $A C' A'$, or $\sin^{-1} \frac{R+r}{a}$, and therefore the arc $D B B'$ equals $R \sin^{-1} \frac{R+r}{a}$. The curved part of the belt on the greater pulley is then $R\left(\pi + 2 \sin^{-1} \frac{R+r}{a}\right)$, and on the smaller pulley it is $r\left(\pi + 2 \sin^{-1} \frac{R+r}{a}\right)$.




$$\therefore\ l = 2\sqrt{a^2 - (R+r)^2} + (R+r)\left(\pi + 2 \sin^{-1} \frac{R+r}{a}\right).$$

From this equation $r$ can be found approximately. For an open belt, $A B$ or $C' E$ is $\sqrt{a^2 - (R - r^2)}$. The angle $B C D = C C' E$, the

sine of which is $\frac{R-r}{a}$. The arc $BD$ equals $R\sin^{-1}\frac{R-r}{a}$, and the arc $AD'$ equals $r\sin^{-1}\frac{R-r}{a}$.

$$\therefore\ l=2\sqrt{a^2-(R-r)^2}+R\left(\pi+2\sin^{-1}\frac{R-r}{a}\right)+r\left(\pi-2\sin^{-1}\frac{R-r}{a}\right)$$

$$=2\sqrt{a^2-(R-r)^2}+(R+r)\pi+2(R-r)\sin^{-1}\frac{R-r}{a}.$$

From this equation another approximate value of $r$ can be found.

Since $R-r$ is small compared with $a$, we have approximately,

$$2\sqrt{a^2-(R-r)^2}=2a-\frac{(R-r)^2}{a},\quad \sin^{-1}\frac{R-r}{a}=\frac{R-r}{a}.$$

Therefore, for an open belt the approximate formula is

$$=2a-\frac{(R-r)^2}{a}+(R+r)\pi+\frac{2(R-r)^2}{a},$$

$$=2a+\frac{(R-r)^2}{a}+(R+r)\pi,$$

from which

$$2r=d=2R-a\pi\pm\sqrt{8(a^2+r^2)-la+a^2\pi^2}.$$

This solution is by O. B. Wheeler.

Solution 2d. If the angle $CC'E=BCD$ be denoted by $\varphi$, then $2AB=2a\cos\varphi$, $BE=R-r=a\sin\varphi$, $BD=R\varphi$, $AD'=r\varphi$.

$$\therefore l=2a\cos\varphi+R\pi+2R\varphi+r\pi-2r\varphi$$

$$=2a\cos\varphi+(R+r)\pi+2(R-r)\varphi$$

$$=2a\cos\varphi+(R+r)\pi+2a\varphi\sin\varphi$$

(1) $$=2a\cos\varphi+2a\varphi\sin\varphi+2R\pi-a\pi\sin\varphi.$$

But $\cos\varphi=1-\frac{1}{2}\varphi^2+\frac{1}{1\cdot2\cdot3\cdot4}\varphi^4-$ &c.,

$\sin\varphi=\varphi-\frac{1}{1\cdot2\cdot3}\varphi^3+\frac{1}{1\cdot2\cdot3\cdot4\cdot5}\varphi^5-$ &c.;

and substituting these values of $\cos\varphi$ and $\sin\varphi$ in (1) we shall find, after reversing the series, that

$$\varphi=-\frac{2}{\pi}\left(\frac{l-2R\pi-2a}{2a}\right)+\frac{4}{\pi^3}\left(\frac{l-2R\pi-2a}{2a}\right)^2-\text{\&c.}$$

But $\quad 2r = d = 2R - 2a\sin\varphi$

$$= D - 2a\left(\varphi - \tfrac{1}{1\cdot2\cdot3}\varphi^3 + \tfrac{1}{1\cdot2\cdot3\cdot4\cdot5}\varphi^5 - \&c.\right).$$

For a simple approximate formula, let $\cos\varphi = 1$, and $\sin\varphi = \varphi$, and (1) becomes

$$l = 2a + 2a\varphi^2 + 2R\pi - a\pi\varphi,$$

$$\therefore \qquad \varphi = \tfrac{1}{4}\pi \pm \sqrt{\tfrac{1}{16}\pi^2 + \frac{l - 2R\pi - 2a}{2a}}.$$

$$\therefore \qquad d = D - 2\left(\tfrac{1}{4}\pi \pm \sqrt{\tfrac{1}{16}\pi^2 + \frac{l - 2R\pi - 2a}{2a}}\right).$$

This solution is by Asher B. Evans.

Joseph Winlock.
Chauncey Wright.
Truman Henry Safford.

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## NOTES AND QUERIES.

1. *Subtraction of Fractions.* Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions; then, as is easily seen,

$$\frac{a}{b} - \frac{c}{d} = \frac{a(d-c) - c(b-a)}{bd} = \frac{ad - bc}{bd}.$$

When $d - c$ is less than $d$, and $b - a$ is less than $b$, the numerator of the second form will be most readily computed; and the advantage will be great when the terms of the fractions are large numbers, but nearly equal to each other. Terquem's *Annales de Mathematique.*

2. It is interesting to notice, that the demonstration of the Pythagorean proposition on page 231 of the Mathematical Monthly is essentially the same as the Indian demonstration contained in the *Bija Ganita*, and referred to as the "figure of the bride's chair," &c. It is involved in the square $AK$ (fig. p. 231). For let $a$ represent the side of the square, and $b$ and $c$ the legs of the equal

right-angled triangles. Then the area of the four equal triangles is $2bc$; and $(b-c)^2$ is the area of the square $SQ$. *Whence $a^2 = 2bc + (b-c)^2 = b^2 + c^2$. This demonstration is given by Dr. HUTTON (Tracts, London, 1812, 3 Vols. 8vo), in his History of Algebra, where may be found a very complete description of the *Bija Ganita* and *Lilawati*. Dr. HUTTON had in his possession Persian MSS. containing translations of both these works. — H. W. RICHARDSON, Waterville College, Maine.

3. *Problem.* A man left 17 horses to be divided among his three sons, the first to have $\frac{1}{2}$, the second $\frac{1}{3}$, and the youngest $\frac{1}{4}$ of the number. They could not agree as to the division, because it required some of the horses to be divided; thus, $\frac{1}{2}$ of $17 = 8\frac{1}{2}$, $\frac{1}{3}$ of $17 = 5\frac{2}{3}$, $\frac{1}{9}$ of $17 = 1\frac{8}{9}$. The sum of the three shares was $16\frac{1}{18}$ horses, and the whole number was not distributed. They carried the case to a judge, who told them, that, if they would abide by his decision, he would give each more than his share, and each should have a whole number of horses. Accordingly, he brought his own horse from the stall, and put him with the 17 others. He then gave $\frac{1}{2}$ of $18 = 9$ horses to the first; $\frac{1}{3}$ of $18 = 6$ to the second; and $\frac{1}{9}$ of $18 = 2$ to the third, making 17 in all. He then returned his own horse to the stall, and left the sons well satisfied. Was the decision just? * * *

4. A correspondent sends us the equation $2\pi\sqrt{-1} = 0$, and proves it as follows. In exponentials,

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* This demonstration has been sent us by JOHN M. BROWN, Esq., of Frankfort, Ky. We have also received several others, which are filed for publication. Prof. ALPHEUS CROSBY, Principal of the Normal School at Salem, Mass., in his work on Geometry, refers to a *Treatise on the Pythagorean Proposition* by HOFFMAN, published at Mayence in 1819, which contains thirty-three different demonstrations of this celebrated theorem. We intend, as soon as we can get a copy of the work, to give in the MONTHLY a brief outline of each demonstration. But if any of our readers have the work, we shall be much obliged for such a synopsis as we propose.

$$\sin x = \frac{1}{2\sqrt{-1}}\left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right).$$

Make $x = \pi$; then $\sin \pi = 0 = \frac{1}{2\sqrt{-1}}\left(e^{\pi\sqrt{-1}} - e^{-\pi\sqrt{-1}}\right)$; or $e^{\pi\sqrt{-1}} = e^{-\pi\sqrt{-1}}$. Multiplying both members by $e^{-\pi\sqrt{-1}}$ we get $e^{2\pi\sqrt{-1}} = e^{\circ} = 1$. $\therefore 2\pi\sqrt{-1} = 0$.

The last conclusion is not correct; for if it were, some one of the factors of $2\pi\sqrt{-1}$ must be zero, which is not the case.

Make $x = n\pi$; then $e^{n\pi\sqrt{-1}} = e^{-n\pi\sqrt{-1}} = \frac{1}{e^{n\pi\sqrt{-1}}}$. Therefore, $e^{2n\pi\sqrt{-1}} = 1$. $\therefore \log 1 = 2n\pi\sqrt{-1}$. And since this is true for all values of $n$, it follows that $\log 1 = 0$, for $n = 0$, but is imaginary for all other values of $n$; that is, log 1 has an infinite number of values, only one of which is real.

It is true in general, that any positive number has one real, and an infinite number of imaginary, logarithms. Let $x$ be the Napierian logarithm of $N$. Then $Ne^{2n\pi\sqrt{-1}} = N = e^x$. Therefore, $\log N + 2n\pi\sqrt{-1} = x$; or $\log N = x - 2n\pi\sqrt{-1}$, which is real only when $n = 0$.

5. *The least Common Multiple.* Some of the arithmetics say, divide the given numbers by *any* number which will divide two or more of them without a remainder, next divide the quotients and undivided numbers by *any* number, &c.; others say, divide by *any prime* number. Will both rules give the correct answer?

6. What is the origin of the term, *Pons Asinorum*, as applied to the fifth proposition of the first book of Euclid?

7. *Twenty-Two Systems of Coördinates.* The usual right-line, or Cartesian coördinates, are $x, y$; the polar $r, \varphi$; the directions of the normal and tangent, or the angles they make with an assumed axis, are $\nu$ and $\tau$; $s$ is the length of the curve; $\varrho$ is its radius

of curvature; $\varepsilon$ the angle between the radius vector and tangent.

$(1)\ y=F(x)$. $(2)\ r=F(\varphi)$. $(3)\ \varrho=F(r)$. $(4)\ \tau=F(s)$.
$(5)\ \varrho=F(s)$. $(6)\ \varepsilon=F(\varphi)$. $(7)\ \tau=F(\varphi)$. $(8)\ r=F(x)$.
$(9)\ x=F(\varphi)$. $(10)\ x=F(\varepsilon)$. $(11)\ \tau=F(x)$. $(12)\ \varrho=F(x)$.
$(13)\ x=F(s)$. $(14)\ \varepsilon=F(r)$. $(15)\ \tau=F(r)$. $(16)\ \varrho=F(r)$.
$(17)\ r=F(s)$. $(18)\ \varrho=F(\varphi)$. $(19)\ \varphi=F(s)$. $(20)\ r=F(\varepsilon)$.
$(21)\ \varrho=F(\varepsilon)$. $(22)\ \varepsilon=F(s)$. The student will find it an excellent exercise to express some familiar curve in as many of these systems as possible. [See Paper on this subject by Rev. THOMAS HILL, in Proc. Am. As. Ad. Sc. 12th meeting, p. 1.]

## ANOTHER SOLUTION OF PRIZE PROBLEM II., No. IV.

By GEORGE EASTWOOD, Saxonville, Mass.

I. IN solving this problem, I shall use Professor GUDERMANN'S method of Spherical Rectangular Coördinates, on account of the remarkable analogy which it exhibits between the properties of lines drawn on the surface of a sphere, and those of lines drawn in a plane. I shall assume, as Mr. MERRILL does in his solution, (No. VIII.), that $ABC$ is the proposed triangle, $AB$ the given base, $P$ its pole, $PO$ a prime meridian passing through the middle of the base, and $PCD$ another meridian passing through the vertex $C$; that the side $AC$ intersects $PO$ in $E$, and that $BC$ meets it in $F$.

Let the vertex $C$ be projected on $PO$ in the point $G$, and put $AO=OB=\alpha$, $OE=\beta$, $OF=\beta'$, $OD=x$, and $OG=y$. Then, if we agree to represent the trigonometric tangents of the coördi-

nate arcs, by the symbols of those arcs the equation of the side $AC$ will be defined by

(1) $$y = \frac{\beta}{\alpha} x + \beta,$$

and of $BC$ by

(2) $$y = -\frac{\beta'}{\alpha} x + \beta'.$$

By spherics,

$$\tan A = \frac{\beta \sec \alpha}{\alpha},$$

(3) $$= \frac{y \sec \alpha}{\alpha + x}, \text{ by reason (1).}$$

$$\tan B = \frac{\beta' \sec \alpha}{\alpha},$$

(4) $$= \frac{y \sec \alpha}{\alpha - x}, \text{ by reason of (2).}$$

But $$\frac{(3)}{(4)} = \frac{\alpha - x}{\alpha + x}$$

is, by the question, a given ratio $= m$, suppose.

(5) $$\therefore x = \frac{(1 - m)\alpha}{1 - m};$$

that is, the vertex of the triangle is always on the meridian circle $PCD$.

From (5) we have, since $x$ and $\alpha$ are tangents,

$$\sin x = \frac{\pm (1 - m) \sin \alpha}{[(m + 1)^2 \cos^2 \alpha + (1 - m)^2 \sin^2 \alpha]^{\frac{1}{2}}},$$

$$\cos x = \frac{\pm (m + 1) \cos \alpha}{[(m + 1)^2 \cos^2 \alpha + (1 - m)^2 \sin^2 \alpha]^{\frac{1}{2}}}.$$

Hence $$\sin AD = \sin (\alpha + x),$$

$$= \frac{\sin 2\alpha}{(m^2 + 2m \cos 2\alpha + 1)^{\frac{1}{2}}}.$$

If in this equation we make $2\alpha = a$, we shall have exactly the same result that Mr. Merrill obtains in his solution, page 259.

II. By the same method of coördinates we shall find that Prize Problem II., No. V., is susceptible of a very neat and simple solution. For, if we designate the required coördinates by $x$ and $y$, the given

points by $x' y'$ and $x'' y''$, and the intercepts of the axes of reference by $\alpha$ and $\beta$; then, by known properties* of great circles arcs, we have

$$(1) \qquad x = -\frac{1}{\alpha}, \quad y = -\frac{1}{\beta},$$

$$(2) \qquad y' = -\frac{\beta}{\alpha} x' + \beta,$$

$$(3) \qquad y'' = -\frac{\beta}{\alpha} x'' + \beta.$$

The elimination of $\alpha$ and $\beta$ from (1), by means of (2) and (3), will obviously satisfy the required conditions of the problem. But the elegant solution of the problem by Mr. OSBORNE in the last MONTHLY, page 292, by an entirely different method, would seem to render further remarks unnecessary.

## THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 307.]

### SECTION IV.

#### ON THE GENERAL MOTIONS AND PRESSURE OF THE ATMOSPHERE.

33. BY the general motions of the atmosphere are meant all those motions produced by a difference of density between the equatorial and polar regions arising principally from a difference of temperature. If the motions of the atmosphere were not resisted by the earth's surface, the results of the preceding sections could be at once applied to them without any modifications, and hence towards the poles there would be a very rapid motion eastward, and in the equatorial

* These properties, and many other analogous ones, of great circle arcs, it is proposed to investigate in subsequent numbers of the MONTHLY.

regions towards the west, and the atmosphere would entirely recede from the poles, and be also depressed about 4,000 feet at the equator, as has been shown in section (2). Although the preceding results, when applied to the atmosphere, are very much modified by the resistances of the earth's surface, yet they will be of great advantage in explaining its general motions; for as there can be no resistance until there is motion, the atmosphere must have a tendency to assume, in some measure, the same motions and figure as in the case of no resistances. Hence, towards the poles the general motions of the atmosphere must be towards the east, and in the torrid zone towards the west; but as these motions, in consequence of the resistances, are small in comparison with those in the case of no resistances, instead of the atmosphere's receding entirely from the poles, as represented in Fig. 1, page 215, there must be only a comparatively small depression there, as represented in Fig. 5, and instead of its being about 4,000 feet lower at the equator than at the place of its maximum height near the tropics (§ 18), there must be only a very slight depression there.

Fig. 5.

34. The force which overcomes the resistance of the earth's surface to the east and the west motions of the atmosphere depends upon the term in the least of our general equations (13) containing

$D_t\theta$ as a factor, which depends upon the interchanging motion of the fluid between the equatorial and the polar regions, and hence the term must vanish at the equator and the poles. All the east or west motion of the atmosphere is consequently destroyed by the resistances at these places, and hence as $D_t\theta$ vanishes there also, there is a belt of calms at the equator, called the equatorial calm belt, and there must be also a region of calms about the poles.

35. As the motion of the atmosphere is east towards the poles and west near the equator, somewhere between the equator and the poles there must be a parallel of no motion east or west, which, in the case of no resistance, was determined upon the hypothesis of an initial state of rest, and found to be at the parallel of 35°, § (18). In the case of the atmosphere this parallel is entirely independent of the initial state of the atmosphere, and depends in a great measure upon the law of resistance, and hence it cannot be accurately determined. It is evident, however, that the east and west motions of the atmosphere at the earth's surface must be such that the sum of the resistances of each part of the earth's surface multiplied into its distance from the axis of rotation, must be equal 0, else the velocity of the earth's rotation would be continually accelerated or retarded, which cannot arise from any mutual action between the surface of the earth and the surrounding atmosphere. Now, as the part of the earth's surface where the motion of the atmosphere is west is much farther from the axis than the part where it is east, the latter part must comprise more than half of the earth's surface, unless the velocity of the eastern motion towards the poles is much greater than that of the western motion near the equator. Therefore, since one-half of the earth's surface is contained between the parallels of 30°, the parallels of no east or west motion at the earth's surface must fall within these parallels, and they are accordingly found to be near the tropics, on the ocean. Hence the maximum height

of the atmosphere, as represented in Fig. (5), must also be near the same parallels.

36. The increase of pressure arising from the accumulation of atmosphere near the tropics, caused principally by the deflecting forces (§ 32) arising from the more rapid east and west motions of the atmosphere in the upper regions, where there is least resistance, gives the atmosphere a tendency to flow from beneath this accumulation both towards the equator and the poles, since the motions, and consequently the forces, which cause this accumulation, are much less near the surface. But on account of the greater density of the atmosphere towards the poles, it has a tendency also to flow, at the earth's surface, from the poles towards the equator. Between the parallels of greatest pressure and the equator, these tendencies combine, and produce a strong surface current, which, combining with the westward motion there, gives rise to the well-known north-east wind in the northern hemisphere, and the south-east wind in the southern hemisphere, called the trade winds. But between the parallels of greatest pressure and the poles, these tendencies are opposed to each other, and the one arising from the accumulation of atmosphere near the tropics being the greater in the middle latitudes, causes the atmosphere to flow at the earth's surface towards the poles; and this motion, combining with the general eastward motion of the atmosphere in those latitudes, gives rise to the south-west wind in the northern hemisphere and the north-west wind in the southern hemisphere, called the passage winds.

37. Near the poles, the tendency to flow towards the equator seems to be the greater, and causes a current there *from* the poles, which, being deflected westward (§ 32), causes a slight north-east wind in the north frigid zone, and a south-east wind in the south frigid zone. But this is only near the earth's surface; and the gener-

al tendency of the atmosphere in the upper regions must be towards the east, as will be seen.

38. Since the atmosphere near the tropics can have no motion in any direction at the earth's surface, there are calm belts there, called the tropical calm belts. Near the polar circles, where the polar and passage winds meet, there must also be calm belts, which may be called polar calm belts. The motions of the atmosphere, therefore, at the earth's surface, if they were not modified by the influence of continents, would be as represented in the interior of Fig. (5), in which the heavy lines represent the calm belts. On account of the influence of the continents, these belts are somewhat displaced and irregular, and on account of the varying position of the Sun, they change their positions a little in different seasons of the year.

The southern limit of the polar winds in the northern hemisphere, and also the limit between the trade and passage winds, has been determined by Prof. J. H. Coffin, from the discussion of a great number of observations at different points, and given in a chart, in his treatise on the winds, published in the 6 seventh volume of the Smithsonian Contributions.

39. That the atmosphere is depressed at the equator and the poles, and has its maximum height near the tropics, as has been represented, is indicated by barometrical pressure. It was formerly thought that this pressure, at the level of the ocean, was very nearly 30 inches in all latitudes; but it is now well established that it is much less towards the poles than near the tropics, and also a little less at the equator. Says Captain Wilkes: "The most remarkable phenomenon which our observations have shown is the irregular outline of the atmosphere surrounding the earth as indicated by the pressure upon the measured column at different parts of the surface. Our barometrical observations show a depression

within the tropics, a bulging in the temperate zone, again undergoing a depression on advancing towards the arctic and antarctic circles." The mean of all the observations, as given in the Report of the Exploring Expedition, from Cape Henry to Madeira, taken between the parallels of 28° and 32°, was 31.215 inches; at Maderia, latitude 32° 53′, 30.176 inches; and in the rainy belt between the parallels of 8° and 12°, 29.987 inches. After passing the equator there was a slight elevation, again reaching its maximum near the tropic of Capricorn. Beyond this there was a gradual depression until about the parallel of 55°, where the barometer was rapidly depressed below 29 inches. After doubling Cape Horn and proceeding towards the equator, the height of the barometer gradually increased again to its usual height in the middle and equatorial latitudes. On sailing south again, in the Pacific Ocean, a depression of the barometer was again observed. The mean of all the observations taken on 22 days, in sailing from Callao to Tahiti, between the parallels of 10° and 15°, was 30.109 inches; and of those made on 32 days, between the parallels of 15° and 20°, was 30.147 inches. The mean of the observations made on 5 days, after leaving Sydney, between the parallels of 35° and 45°, was 30.305 inches; of those made on 7 days, between the parallels of 45° and 55°, was 29.790 inches; of those taken on 8 days, between the parallels of 55° and 65°, was 29.378 inches. The mean also of all those taken along the antarctic continent was 29.040 inches.

40. Says Sir JAMES ROSS (*Voyage to the Southern Seas*, Vol. 2, p. 383): "Our barometrical experiments appear to prove that the atmospheric pressure is considerably less at the equator than near the tropics; and to the south of the tropic of Capricorn, where it is greatest, a gradual diminution occurs as the latitude is increased, as will be shown from the following Table, derived from hourly observations of the height of the column of mer-

cury between the 20th of November, 1839, and the 31st of July, 1843."

EXTRACT FROM ROSS'S TABLE.

| LATITUDE. | PRESSURE. | LATITUDE. | PRESSURE. | LATITUDE. | PRESSURE. |
|---|---|---|---|---|---|
| | Inches. | | Inches. | | Inches. |
| Equator, | 29.974 | 42° 53′ | 29.950 | 55° 52′ | 29.360 |
| 13° 0′ S. | 30.016 | 45 0 | 29.664 | 60 0 | 29.114 |
| 22 17 | 30.085 | 49 8 | 29.467 | 66 0 | 29.078 |
| 34 48 | 30.023 | 51 33 | 29.497 | 74 0 | 28.928 |
| | | 54 26 | 29.347 | | |

41. The following table, first published by M. Schouw, and reduced here from millimetres to English inches, shows that there is a similar bulging of the atmosphere in the middle latitudes, and depression at the pole in the northern hemisphere, as has been observed in the southern hemisphere.

| PLACE. | LATITUDE. | PRESSURE. | PLACE. | LATITUDE. | PRESSURE. |
|---|---|---|---|---|---|
| | | Inches. | | | Inches. |
| Cape, | 33° 0′ S. | 30.040 | London, | 51° 30′ | 29.961 |
| Rio Janeiro, | 23 S. | 30.073 | Altona, | 53 30 | 29.937 |
| Christianburg, | 5 30 N. | 29.925 | Dantzic, | 54 30 | 29.925 |
| La Guayra, | 10 | 29.928 | Konigsberg, | 54 30 | 29.941 |
| St. Thomas, | 19 | 29.941 | Apenrade, | 55 | 29.905 |
| Macao, | 23 | 30.039 | Edinburgh, | 56 | 29.851 |
| Teneriffe, | 28 | 30.087 | Christiana, | 60 | 29.866 |
| Madeira, | 32 30 | 30.126 | Bergen, | 60 | 29.703 |
| Tripoli, | 33 | 30.213 | Hardanger, | 60 | 29.700 |
| Palermo, | 38 | 30.036 | Reikiavig, | 64 | 29.607 |
| Naples, | 41 | 30.012 | Godthaab, | 64 | 29.603 |
| Florence, | 43 30 | 29.996 | Eyafiord, | 66 | 29.669 |
| Avignon, | 44 | 30.000 | Godhaven, | 69 | 29.674 |
| Bologna, | 44 30 | 30.008 | Upernavik, | 73 | 29.732 |
| Padua, | 45 | 30.008 | Mellville Isle, | 74 30 | 29.807 |
| Paris, | 49 | 29.976 | Spitzbergen, | 75 30 | 29.795 |

42. From the preceding tables, it is seen that the barometrical pressure is much less, especially in the southern hemisphere towards the poles than at the equator, although the density towards the poles is much greater, and hence the depression there must be considerable.

## ON THE SOLUTION OF EQUATIONS.

By John Borden, Chicago, Illinois.

The general equation of the second degree, $x^2 + Ax + B = 0$, may be solved as follows. Assume $x = -a + b$, $x = -a - b$ as the values of the roots. Then

$$(x + a - b)(x + a + b) = x^2 + 2ax + a^2 - b^2 = x^2 + Ax + B;$$

and therefore $A = 2a$, $B = a^2 - b^2$; or, $a = \frac{A}{2}$, $b = \pm \frac{1}{2}\sqrt{A^2 - 4B}$.

Therefore, $x = -\frac{A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4B}$, as by the usual method.

The general form of the cubic equation is

$$(1) \qquad x^3 + Ax^2 + Bx + C = 0.$$

If we suppose one root known, as $x = -c$, then the quotient obtained by dividing (1) by $x + c$ must equal zero; that is,

$$x^2 + (A - c)x - (A - c)c + B = 0.$$

Therefore the other two roots are

$$(2) \qquad x = -\frac{A - c}{2} \pm \sqrt{A^2 + 2Ac - 4B - 3c^2}.$$

By making $x = x' - \frac{A}{3}$, as in the common transformation, (1) becomes

$$(3) \qquad x'^3 + \left(B - \frac{A^2}{3}\right)x' + C - \frac{AB}{3} + \frac{2A^3}{27} = 0.$$

If, in this equation, $B - \frac{A^2}{3} = 0$, then

(4) $x' = \left(\frac{A^3}{27} - C\right)^{\frac{1}{3}}, \quad x' = -\frac{1}{2}\left(\frac{A^3}{27} - C\right)^{\frac{1}{3}} \pm \frac{1}{2}\left(\frac{A^3}{27} - C\right)^{\frac{1}{3}}\sqrt{-3},$

If the third term of (3) reduces to zero, then

$$x' - 0, \quad x' = \pm\sqrt{\frac{A^2}{3} - B};$$

or lastly, if the third term, as well as the coefficient of $x'$, is zero, then all the values of $x'$ are zero, and the values of $x$ in (1) all equal $-\frac{A}{3}$. But, none of these cases occurring, (3) is of the general form of (1), if $A = 0$; hence if one root $= -c$, the expression for the other two becomes

(5) $$x' = \frac{c}{2} \pm \sqrt{-4B' - 3c^2}.$$

Therefore, the general form of the roots of (3) are

(6) $$x' = -c, \quad x' = \frac{c}{2} + d, \quad x' = \frac{c}{2} - d;$$

and those of (1), since $x = x' - \frac{A}{3}$, are

(7) $$x = -\frac{A}{3} - c, \quad x = -\frac{A}{3} + \frac{c}{2} + d, \quad x = -\frac{A}{3} + \frac{c}{2} - d.$$

The difficulty, which arises in obtaining the values of $c$ and $d$ in terms of the coefficients appears to come from the fact, that any thing which can be predicated as true of one of the roots in general terms is also true of all the others.

There is one other transformation, of which an equation of the form $x^3 + Bx + C = 0$, for instance, is susceptible. If $x' = xy$, then $x'^3 + Bx'y^2 + Cy^3 = 0$, and we may assume $By^2 = Cy^3$; or $By^2 = E$; or $Cy^3 = d$; or, lastly, $x' = x, y = 1$, in which case $y = \frac{1}{x}$, and the equation becomes $y^3 + \frac{B}{C}y^2 + \frac{1}{C} = 0$, in which the roots, or values of $y$, are the reciprocals of the values of $x$, and the solution of one equation involves the solution of the other.

By combining equations (6) there results

$$(8)\qquad x^3-\left(\frac{3\,c^2}{4}+d^2\right)x+\frac{c^3}{4}-c\,d^2=0.$$

From this it appears, that, unless $d$ is imaginary, and $d^2>\frac{3\,c^2}{4}$, the coefficient of $x$ is essentially negative; and as the third term is the product of all the roots with their signs changed, and the two minor roots are of the same sign, and the maximum root of the contrary sign, that the third term and the maximum root will have contrary signs. If the coefficients of (8) be equated with those of

$$(9)\qquad x^3+B\,x+C=0,$$

and $c$ be eliminated, the resulting equation is

$$(10)\qquad d^6+\tfrac{3}{2}\,B\,d^4+\tfrac{9}{16}\,B^2\,d^2+\tfrac{1}{16}\,B^3+\tfrac{27}{64}\,C^2=0; \text{ or}$$

$$(11)\qquad d^3+\tfrac{3}{4}\,B\,d\pm\sqrt{-\tfrac{1}{16}\,B^3-\tfrac{27}{64}\,C^2}=0.$$

If the third term of (11) be equal to zero, then $d=0$, and two of the roots of (8) are equal, as will appear from (6). But if $d=0$, then the third term of (11) equals zero, and expresses the relative values of $B$ and $C$ in such case; namely, $4\,B^3=-27\,C^2$. It also follows, as the third term of (8) equals the third term of (9), that $c=(-4\,C)^{\frac{1}{3}}$. And this is the value of the maximum root when the two minor roots are equal. Further, equation (10) shows that $d$ may have six values, or two for each value of $c$ in (6). And upon investigation, it will be found that $d$ equals one half the algebraic difference of the roots of (9), taken two in a set. And since the two minor roots of (9) have the same sign, the values of $d$ are equal one half their arithmetical difference, and at the same time the maximum root of (9) is equal to their arithmetical sum.

By reference to equation (10), it will appear, that, although it is of the sixth degree, yet in form it is of the third; and is composed of the two cognate factors represented in (11). And this leads to a consideration of the cognate factors into which any equation may be

decomposed. For, suppose the general cubic equation to be put under the form

$$(12)\quad (x^2-a^2)(x^2-b^2)(x^2-c^2)=x^6-A'x^4+B'x^2-C'=0,$$

and assume

$$(x^3+Ax^2+Bx+C)(x^3-Ax^2+Bx-C)=0$$

as two of the cubic factors into which it may be decomposed. By multiplying we obtain

$$(13)\quad \begin{array}{l} x^6-(A^2-2B)x^4+(B^2-2AC)x^2-C^2=0=(12) \\ x^6-\qquad\qquad A'x^4+\qquad\qquad B'x^2-C'=0; \end{array}$$

and by equating the coefficients we have

$$(14)\qquad A^2-2B=A',\quad B^2-2AC=B',\quad C^2=C';$$

and by eliminating to find $A$ we get

$$(15)\qquad A^4-2A'A^2-8A\sqrt{C'}+A'^2-4B'=0,$$

which is an equation of the fourth degree, wanting its second term. But (12) is of the form

$$(x^2-a^2)(x^2-b^2)(x^2-c^2)=0,$$

and equations (13) are respectively of the form

$$(x+a)(x+b)(x+c)=0,\quad (x-a)(x-b)(x-c)=0.$$

Therefore, $A$ is equal to the sum of the square roots of the roots of (12), considered as an equation of the third degree. Therefore, the roots of an equation of the form of (15) are determined; and the bi-quadratics are solved. Or, by eliminating $A$ from equations (14), we obtain

$$(16)\qquad B^4-2B'B^2-8C'B+B'^2-4A'C'=0,$$

which is also an equation of the fourth degree, wanting its second term, and this furnishes another solution of bi-quadratics. For $B$ is equal to the sum of the products of the square roots of the roots or

(12) combined, two in a set, (12) being taken as an equation of the third degree.

By examining the cognate quadratic factors of an equation of the second degree, a relation of the like kind is established between equations of the second and third degrees, as follows:

(17) $$(x^3-a^3)(x^3-b^3)=0=x^6-A'x^3+B',$$

(18) $(x-a)(x-b)=0=x^2-Ax+B$, the factor sought.

$$(x^3-a^3)=(x-a)(x+\tfrac{a}{2}\pm\tfrac{a}{2}\sqrt{-3})\,;\; x^3-b^3=(x-b)(x+\tfrac{b}{2}\pm\tfrac{b}{2}\sqrt{-3}).$$

(19) $$\left(x+\tfrac{a}{2}+\tfrac{a}{2}\sqrt{-3}\right)\left(x+\tfrac{b}{2}+\tfrac{b}{2}\sqrt{-3}\right) = x^2+\left(\tfrac{A}{2}+\tfrac{A}{2}\sqrt{-3}\right)x-\tfrac{B}{2}+\tfrac{B}{2}\sqrt{-3}.$$

(20) $$\left(x+\tfrac{a}{2}-\tfrac{a}{2}\sqrt{-3}\right)\left(x+\tfrac{b}{2}-\tfrac{b}{2}\sqrt{-3}\right) = x^2+\left(\tfrac{A}{2}-\tfrac{A}{2}\sqrt{-3}\right)x-\tfrac{B}{2}-\tfrac{B}{2}\sqrt{-3}.$$

But the right-hand members of equations (18), (19), (20) multiplied together, give (17).

$$\therefore \quad x^6-(A^3-3AB)x^3+B^3=x^6-A'x^3+B'=0.$$

(21) Whence $A^3-3AB=A'$, $B^3=B'$, in which $a^3+b^3=A'$, $a^3b^3=B'$, and $a+b=A$, $ab=B$. Whence

(22) $$A=\left(\tfrac{A'}{2}+\tfrac{1}{2}\sqrt{A'-4B'}\right)^{\frac{1}{3}}+\left(\tfrac{A'}{2}-\tfrac{1}{2}\sqrt{A'-4B'}\right)^{\frac{1}{3}}.$$

Therefore, if $x^3+px=q$ be identical with (21), then $B'=\frac{1}{27}p^3$, $A'=q$, and

$$x=\left(\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}\right)^{\frac{1}{3}}+\left(\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

which is the well known Cardan formula.

If in (17), $(x^3+a^3)\ (x^3+b^3)$ had been taken, the same result would have been obtained.

If in (17), $(x^3+a^3)(x^3-b^3)$ had been taken, then

(23) $$A^3+3AB=A'$$

would result, which corresponds to $x^3-px=q$, and

(24) $$x=\left(\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}+\left(\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

Equations (22) and (24) are the same as $A=a+b$. And from the values of $a$ and $b$, the values of $\frac{a}{2}\pm\frac{a}{2}\sqrt{-3}, \frac{b}{2}\pm\frac{b}{2}\sqrt{-3}$ can be obtained; also, from the various combinations of the six values which all enter into (17), its different quadratic factors might be constructed. If $B'$ in that equation be supposed to arise from $B.B.B$, then $A$ must correspond to such a combination of the values. Now in such case the combination is

$$(-a)(-b);\ \left(\frac{a}{2}+\frac{a}{2}\sqrt{-3}\right)\left(\frac{b}{2}-\frac{b}{2}\sqrt{-3}\right);$$
$$\left(\frac{a}{2}-\frac{a}{2}\sqrt{-3}\right)\left(\frac{b}{2}+\frac{b}{2}\sqrt{-3}\right).$$

And from such a combination the corresponding values of $A$ can be constructed; and these are the different values of $A$ or $x$ in (21) and (23), and correspond with the CARDAN formulas. By comparing the values of $A$ as thus constructed with the forms as given in (5), we find, by reduction, that the roots of an equation of the form $x^3+Bx+C=0$, (25) are

(26) $$x=(a+b),\ x=-\frac{a+b}{2}+\frac{a-b}{2}\sqrt{-3},\ x=-\frac{a+b}{2}-\frac{a-b}{2}\sqrt{-3},$$

in which

$$a=\left(-\frac{C}{2}+\sqrt{\frac{C^2}{2}+\frac{B^3}{27}}\right)^{\frac{1}{3}},\ b=\left(-\frac{C}{2}-\sqrt{\frac{C^2}{4}+\frac{B^3}{27}}\right)^{\frac{1}{3}}.$$

If the roots are all real, it is evident that $a-b$ must be imaginary. But this occurs when $B$ is negative, and $\frac{B^3}{27}>\frac{C^2}{4}$, and the values of $a$ and $b$ are not then obtainable by a direct reduction, and the equation is then said to belong to the irreducible case. The method of

obtaining the values of $a$ and $b$ in such case, is by a development of their values. If

$$(27) \qquad m = -\frac{4B^3 + 27C^2}{C^2},$$

the value of $x$ in (25) can be put under the form

$$(28) \quad x = (-4C)^{\frac{1}{3}}\left(1 - \frac{1}{q} m^{\frac{1}{2}} \sqrt{-3}\right)^{\frac{1}{3}} + (-4C)^{\frac{1}{3}}\left(1 + \frac{1}{q} m^{\frac{1}{2}} \sqrt{-3}\right)^{\frac{1}{3}},$$

or by developing

$$(29) \quad x = (-4C)^{\frac{1}{3}}\Big(1 + \frac{1}{3^5} m - \frac{10}{3^{11}} m^2 + \frac{154}{3^{17}} m^3 - \frac{935}{3^{22}} m^4 + \frac{11.13.17.23}{3^{29}} m^5 - \frac{8.13.17.23.29}{3^{35}} m^6 + \frac{17.20.23.29.38}{3^{40}} m^7 - \&c.\Big);$$

or,

$$(30) \quad x = (-4C)^{\frac{1}{3}}\Big(1 + \frac{m}{\log^{-1} 2.385606} - \frac{m^2}{\log^{-1} 4.247141} + \frac{m^3}{\log^{-1} 5.921154} - \frac{m^4}{\log^{-1} 7.523670} + \frac{m^5}{\log^{-1} 9.085424} - \frac{m^6}{\log^{-1} 10.622890} + \frac{m^7}{\log^{-1} 12.143496} - \&c.\Big).$$

Equations (29) and (30) appear to give the value of the maximum root, and these series are rapidly converging, when $m$ is a small number. If $m = 0$, the two minor roots are equal, and the maximum root equals $(-4C)^{\frac{1}{3}}$. If both terms of the values of $x$ in (28) be developed in the descending powers of the imaginary, the values of $x$ become

$$(31) \quad x = -(-4C)^{\frac{1}{3}}\Big(m^{-\frac{1}{3}} - 5m^{-\frac{4}{3}} + 66\, m^{-\frac{7}{3}} - 17.66 m^{-\frac{10}{3}} + 17.23.55\, m^{-\frac{13}{3}} - 17.23.29.39 m^{-\frac{16}{3}} + 17.23.24.29.35 m^{-\frac{19}{3}} - \&c.\Big).$$

This equation appears to give the minimum root, and will be converging when $m$ is a large number. If one term of the value of $x$ in (28) be dropped in the ascending powers of the imaginary, and the other in the descending, then the terms involving the imaginary cannot cancel each other, for the sum of the real terms in that case equals the half sum of (29) and (30), which is not a root of (25). Nor are we to suppose that the sum and difference of any number

of imaginary terms are equal to a real number. Therefore the expression for the value of $x$ in this case is part real and part imaginary. But in the irreducible case now under consideration, all the roots are real; therefore this value of $x$ is not a root of (25), but is still to be found among the combinations of the six roots or values which go to make up the quadratic factor of (17). If the values of $x$ in (26) be combined, there results

(32) $$x^3 - 3\,a\,b\,x - a^3 - b^3 = 0,$$

from which, by comparing with $x^3 + B\,x + C = 0$, we have

(33) $$-3\,a\,b = B, \text{ or } -27\,a^3\,b^3 = B^3;\ -a^3 - b^3 = C.$$

But it is evident, that, if $-\frac{a}{2} \pm \frac{a}{2}\sqrt{-3}$, $-\frac{b}{2} \pm \frac{b}{2}\sqrt{-3}$ were substituted in the first set of (33), $-a^3 - b^3 = C$ would be the same. The same would be the case if $-\frac{B}{2} \pm \frac{B}{2}\sqrt{-3}$ were substituted. Therefore the values of $a$ and $b$ derived therefrom involve these various cases, and the equations

(34) $$x^3 + B\,x + C = 0,\ x^3 + \left(-\frac{B}{2} + \frac{B}{2}\sqrt{-3}\right)x + C = 0,$$
$$x^3 + \left(-\frac{B}{2} - \frac{B}{2}\sqrt{-3}\right) + C = 0$$

are all equally solved by the Cardan formulas.

It results from these developments, that an expression of the form $(m+n)^{\frac{1}{3}}$ does not give always the same value, when developed according to the ascending powers of $n$, as when developed according to the ascending powers of $m$.

If an equation of the fourth degree be examined with reference to its quadratic factors; as, for instance, if

(35) $$(x^2 + Ax + B)(x^2 + Cx + D) = x^4 + ax^3 + bx^2 + cx + d = 0,$$

and the four roots are $a', b', c', d'$, it is plain that $B$, $C$, $D$, or $A$ will have six values. For the values of $B$ are $a'b'$, $a'c'$, $a'd'$, $b'c'$, $b'd'$, $c'd'$.

Thence the value of any one of the coefficients of either of the quadratic factors, when obtained, should be expressed in the terms of an equation of the sixth degree. By equating, we get

(36) $A + C = a, \; AC + B + D = b, \; BC + AD = c, \; BD = d.$

By eliminating to find the value of $B$, we have, after reduction,

$$(37) \quad B^6 - bB^5 + (ac - d)B^4 + (2bd - c^2 - a^2 d)B^3 + (acd - d^2)B^2 - bd^2B + d^3 = 0.$$

Hence, assuming

$$(38) \quad x^6 + fx^5 + gx^4 + hx^3 + (gl^{\frac{1}{3}} - 2l^{\frac{2}{3}})x^2 + fl^{\frac{2}{3}}x + l = 0,$$

and equating the coefficients of (37) and (38), the values of $a, b, c, d$, the coefficients of (35), can easily be obtained. Hence the roots of an equation of the sixth degree of the form (38), as also its reciprocal, with their various modifications, are dependent upon the roots of the general bi-quadratic (35); for the roots of (38) are the roots of (35) multiplied together, two in a set.

This method of examining an equation by its factors, furnishes a general method of solution of equations; the only objection being, that, as we reason from the less degree, we are bound to accept the greater degree, in whatever form the equation may present itself. And if any equation be taken as the subject of examination, the question put is, What are the equations of the higher degrees, whose forms depend for solution upon the equation under examination? It may be said, that there are certain forms of equations of all degrees above the second, which may be evolved from equations of the second degree, and their solution thereby obtained. As, for instance, the equation of the fifth degree of the form

$$(39) \qquad x^5 + Bx^3 + \frac{B}{5}x + C = 0$$

has for the value of $x$,

$$(40) \quad x = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \left(\frac{B}{5}\right)^5}\right)^{\frac{1}{5}} + \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \left(\frac{B}{5}\right)^5}\right)^{\frac{1}{5}};$$

and its reciprocal,

$$(41) \qquad x^5 + \frac{B^2}{56} x^4 + \frac{B}{C} x^2 + \frac{1}{C} = 0,$$

is therefore solved also.

Also, the form of the equation of the seventh degree is

$$(42) \qquad x^7 + B x^5 + \tfrac{2}{7} B^2 x^3 + \tfrac{1}{49} B^3 x + C = 0,$$

its solution is

$$(43) \quad x = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \left(\frac{B}{7}\right)^7}\right)^{\frac{1}{7}} + \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \left(\frac{B}{8}\right)^7}\right)^{\frac{1}{7}};$$

its reciprocal is

$$(44) \qquad x^7 + \frac{B^3}{C} x^6 + \tfrac{2}{7} B^2 x^4 + \frac{B}{49\,C} x^2 + \frac{1}{C} = 0.$$

In all these cases, there are only two arbitrary quantities, $B$ and $C$, from which to determine the coefficients; but by the introduction of the transformation $x' = y\,x$, the coefficients may be variously modified. The square root of an equation of the fifth degree of the general form will probably evolve an equation of the eighth degree; and if so, it will lack but one arbitrary quantity, with the assistance of the transformations, of showing the dependence of the general equation of the eighth degree upon one of the fifth.

If an equation of the third degree of the form

$$(45) \qquad x^{15} - A' x^{10} + B' x^5 - C' = 0$$

be resolved into its five factors, by the method as above, we obtain

$$(46) \qquad A^5 - 5\,B\,A^3 + 5\,C A^2 + 5\,B^2 A - 5\,B\,C = A';$$

$$(47) \qquad B^5 - 5\,A\,C B^3 + 5\,C^2 B^2 + 5\,A^2 C^2 B - 5\,A\,C^3 = B';$$

$$(48) \qquad C^5 = C',$$

from which it will require some industry to eliminate $A, B, C$.

There is but one other matter to be named; and that is as to the question, How are the imaginary roots to be interpreted? In the quadratic equation, when the roots become imaginary, the conditions are said to be impossible, and they are used to detect such

impossible conditions. But in equations of a degree above the second, they may be part real and part imaginary; and in such case, how is the test to be used? For in such case, the imaginary roots are as truly roots of the equation as the real roots. There is but little doubt that they admit of geometrical interpretation. Otherwise, equations of the higher orders, instead of being of extraordinary power, from their extreme flexibility and great capacity, may be said to be labyrinths filled with nests of ghostly quantities.

## Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the May number of the MONTHLY: —

C. M. WOODWARD, Junior Class, Harvard College, answered all the questions. (BENJAMIN PEIRCE, Prof.)

HORACE OTIS, Adams Centre, N. Y., answered I. and IV.

P. BARTON, Amsterdam, N. Y., answered all but III.

ROLAND THOMPSON, Junior Class, Jefferson College, Cannonsburg, Penn., answered I. and IV. (JOHN FRASER, Prof.)

DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y., answered all the questions.

W. F. OSBORNE, Sophomore Class, Wesleyan University, Middletown, Conn., answered all the questions. (J. M. VANVLECK, Prof.)

ASHER B. EVANS, Madison University, Hamilton, N. Y., answered all the questions.

GUSTAVUS FRANKENSTEIN, Springfield, Ohio, answered all the questions.

GEORGE A. OSBORNE, Jr., Lawrence Scientific School, answered all the questions. (H. L. EUSTIS, Prof.)

GEORGE W. JONES, Jr., Senior Class, Yale College, answered all the questions. (H. A. NEWTON, Prof.)

W. MURRAY STERLING, Student, Baltimore, Md., answered questions I., II. and IV. (TIMOTHY CRIMMIN, Teacher.)

A STUDENT, High School, Baltimore, Md., answered questions I., II. and IV. (JAMES MCINTIRE, Prof.)

It gives us pleasure to add the following names to our list of coöperators and contributors: EDWIN HAAS, Burlington, N. J.; R. C. MATTHEWSON, San Francisco, Cal.; W. C.

DENNIS, Key West, Florida; AUGUST SONNTAG, Acting Assistant at the Dudley Observatory, Albany, N. Y.; WILLIAM J. LEWIS, San Francisco, Cal. . . . . To the following Card, which we are permitted to lay before the readers and friends of the MATHEMATICAL MONTHLY, we have nothing to add, except that it amply repays us for the care and labor we have already devoted to the work. All we ask, is such a support as shall enable us to publish promptly all we receive worthy of being put in more permanent form:—

"*Boston*, May 25th, 1859.

"The undersigned, having watched with great interest the establishment of a new MATHEMATICAL JOURNAL at Cambridge, under the editorship of Mr. J. D. RUNKLE, are desirous of calling the attention of the patrons of sound learning to this work. The 'MATHEMATICAL MONTHLY' commenced its existence in October, 1858, and has been conducted on a plan that cannot fail to make it an instrument of great good. It is addressed to students as well as to professors; and has doubtless already given a new impulse to mathematical studies, wherever it has been introduced. A work of this kind should not be suffered simply to live. It has now about eleven hundred subscribers; and is not, perhaps, likely to be altogether discontinued, while its present support remains. But its friends should not be satisfied with this. A liberal subscription should insure to it a vigorous, energetic, long-continued life; and should enable it not only to preserve its present excellent form, and the stimulus of its prizes,—now amounting to about three hundred dollars,—but to make new improvements; to increase its size, without increasing its terms; to secure the best matter, by paying contributors, if necessary; and to compensate its editor, in part at least, for the time he bestows upon it.

"This appeal is made from no suggestion of the editor, but from the conviction that the work deserves a wide patronage, and must secure it, in order to be *permanently* successful. The circle of strictly mathematical readers in this country is yet small. To enlarge it, nothing can be more surely relied on than a well-conducted Journal; but until this is done, and the work is thus made to support itself, the aid of *all* true friends of Science must be invoked.

"JAMES WALKER.
JARED SPARKS.
BENJAMIN PEIRCE.
JOSEPH LOVERING.
G. P. BOND.
JOSIAH QUINCY.
EDWARD EVERETT.
J. INGERSOLL BOWDITCH."

We hope that Prof. WILLIAM RUTHERFORD, of the Royal Military Academy, Woolwich, England, will pardon us for giving, in connection with the above, an extract from his polite note of May 4th, 1859:—

"I have also got all the numbers of your new Monthly Mathematical publication, with which I am very much pleased. It will be a useful work, and I regret we have not any similar publication in Britain."

Prof. GIBBES has sent us the following errata: On page 338, last line, for $> 0.250$, read $< 0.250$; page 339, line —5, in the value of $y$, put exponent $\frac{1}{2}$ outside the parenthesis. A few unimportant verbal errors we omit, as they will give the reader no difficulty. On page 350, for 63 cents, read 36.

THE

# MATHEMATICAL MONTHLY.

Vol. I....SEPTEMBER, 1859....No. XII.

## REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VIII., Vol. I.

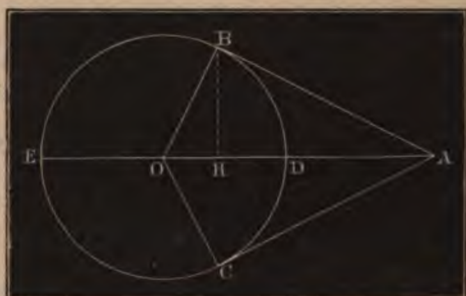
The first Prize is awarded to George A. Osborne, Jr., of the Lawrence Scientific School, Cambridge, Mass.

The second Prize is awarded to George W. Jones, Jr., Senior, Yale College, New Haven, Ct.

### Prize Solution of Problem I.

"If $x$ be the distance of the eye from the centre of a sphere, of which the radius is $r$, prove that the visible part of its surface is to the invisible as $x-r:x+r$."

Let $A$ denote the position of the eye, $O$ the centre of the sphere, and $BDCE$ a great circle of the sphere cut by a plane passing through $A$ and $O$. Draw the tangents $AB$ and $AC$, the radii $OB$ and $OC$, and $BH$ perpendicular to $AE$. Then




$$\frac{AO}{OB}=\frac{OB}{OH} \text{ or, } \frac{x}{r}=\frac{r}{OH}.$$

$$\therefore \frac{x-r}{x+r}=\frac{r-OH}{r+OH}=\frac{DH}{HE}.$$

Since zones of the same sphere are to each other as their alti-

tudes, the visible and invisible portions or zones are to each other as their altitudes $DH : HE :: x - r : x + r$.

This solution is by GEORGE A. OSBORNE, Jr.

PRIZE SOLUTION OF PROBLEM II.

"Transform the series

$$1 + 8 + 19 + 34 + 53 + 76 + \&c.,$$

so as to find the sum of $n$ terms by means of the usual formula for summing the squares of the natural numbers."

Assume

$$A + Bn + Cn^2 + Dn^3 = 1 + 8 + 19 + 34 + 53 + 76 + \ldots.$$

and make $n = 1, 2, 3, 4$, successively. We get

$$A + B + C + D = 1,$$
$$A + 2B + 4C + 8D = 1 + 8,$$
$$A + 3B + 9C + 27D = 1 + 8 + 19,$$
$$A + 4B + 16C + 64D = 1 + 8 + 19 + 34.$$

From these equations we get

$$D = \tfrac{2}{3},\ C = \tfrac{3}{2},\ B = -\tfrac{7}{6},\ A = 0;$$

and, by substitution, the assumed formula becomes

$$\frac{(4n^2 + 9n - 7)n}{6} = \frac{2(2n^2 + 3n + 1)n}{6} + \frac{n(3n - 9)}{6}$$
$$= \frac{n(n+1)(2n+1)}{3} + \frac{(n-3)n}{2},$$

as the sum of the series.

$$\therefore 1 + 8 + 19 + \ldots 2n^2 + n - 2 = 2(1 + 4 + 9 + \ldots n^2) + \frac{n(n-3)}{2}.$$

The part $\frac{n(n-3)}{2}$ is the sum of an arithmetical series, of which the first term is 2, the last term $n - 2$, the number of terms $n - 3$, and the common difference 1. This is also shown by

$$1 + 8 + 19 + 34 + \ldots = 1 + 2.4 + (1 + 2.9)$$
$$+ (2 + 2.16) + (3 + 2.25) + \ldots.$$

$$= 2(1 + 2^2 + 3^2 + 4^2 + \ldots . n^2) + (2 + 3 + 4 + \ldots . n - 2)$$

$$= n\frac{(n+1)(2n+1)}{3} + \frac{(n-3)n}{2}.$$

This solution is by GUSTAVUS FRANKENSTEIN.

PRIZE SOLUTION OF PROBLEM III.

"If $a, b, c$ are the sides of a spherical triangle, and $A, B, C$ the opposite angles, prove that

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a."$$

If we multiply the elementary formulas of spherical trigonometry,

$$\cos a = \sin b \sin c \cos A + \cos b \cos c$$

$$\cos A = \sin B \sin C \cos a - \cos B \cos C,$$

by $\cos A$ and $\cos a$ respectively, we have

$$(1) \qquad \begin{aligned} \cos a \cos A &= \sin b \sin c \cos^2 A + \cos b \cos c \cos A \\ &= \sin B \sin C \cos^2 a - \cos B \cos C \cos a. \end{aligned}$$

Also since $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C},$$

we have $$\frac{\sin b \sin c}{\sin B \sin C} = \frac{\sin^2 a}{\sin^2 A} = \frac{1 - \cos^2 a}{1 - \cos^2 A}.$$

$$(2) \quad \therefore \sin b \sin c (1 - \cos^2 A) = \sin B \sin C (1 - \cos^2 a),$$

Adding (2) to second and third members of (1) gives

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a.$$

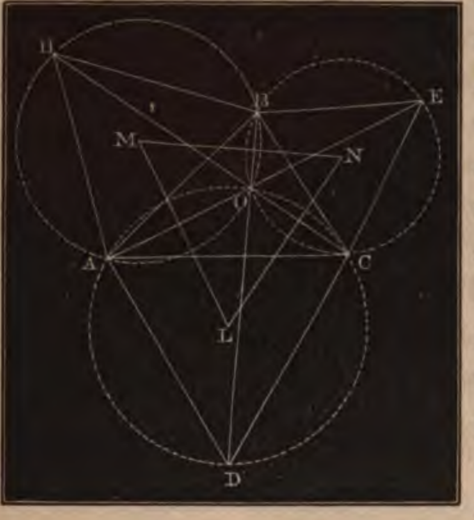
This solution is by GEORGE A. OSBORNE, Jr.

PRIZE SOLUTION OF PROBLEM IV.

"If, on the sides of a given plane triangle, equilateral triangles be constructed, prove that the triangle formed by joining the centres of these equilateral triangles will also be equilateral; also prove that the straight lines joining the vertices of the equilateral triangles and the opposite angles of the given triangle are equal, and all intersect in the same point."

Let $ABC$ be the given triangle. Construct the equilateral triangles $ACD$, $AHB$, and $BEC$, and circumscribe a circle about

each. These circles will intersect at the same point, $O$; for consider $O$ to be the intersection of the two circles $AHB$ and $BEC$, and draw $OA$, $OB$, and $OC$. Since the angles $AHB$ and $BEC$ are each $= 60°$, their supplements $AOB$ and $BOC$ are each $= 120°$. $\therefore$ the angle $AOC$ is also $= 120°$, and is hence the supplement of $ADC$; therefore the circle $ADC$ must pass through the point $O$. Moreover, the lines joining the centres of the equilateral triangles, or of their circumscribing circles, being perpendicular to the common chords $OA$, $OB$, and $OC$, will meet at angles of 60° and form an equilateral triangle.




Draw $OE$, $OD$, and $OH$, then the angle $EOC = EBC$, which is the supplement of $COA$; therefore $OA$ and $OE$ form one right line. Similarly it may be shown that $HOC$ and $BOD$ are right lines, which, together with $AOE$, intersect in a common point $O$.

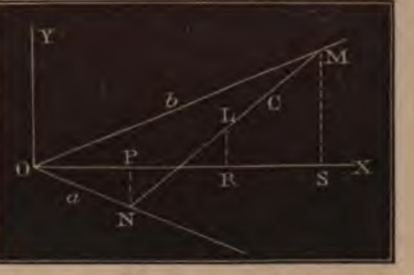
Since the two angles $BCD$ and $ECA$ are equal and included between equal sides, the triangles $BCD$ and $ECA$ are equal and $AE = BD$; similarly it may be shown that $BD = HC$.

This solution is by George A. Osborne, Jr.

### Prize Solution of Problem V.

"If an angle ($A$), and the sum of the squares of the sides of a plane triangle, be given ($= 8a^2$), prove that the curve which continually bisects the side opposite to $A$ is an ellipse, and determine the numerical values of its principal diameters when $A = 60°$, and $a = 10$."

Let $M\,O\,N$ be the given angle $A$, and $M\,N$ one of the positions of the opposite side. Denote by $b$, $c$, and $d$ the sides $O\,M$, $M\,N$, and $N\,O$ respectively. Draw $O\,X$ bisecting the angle $A$, and to this, as the coördinate axis of $X$ and the vertex $O$ as origin, let the required locus be referred. $L$, the middle point of $M\,N$, is a point of this locus. Draw the perpendiculars $N\,P$, $L\,R$, and $M\,S$; then $O\,R=x$, and $L\,R=y$,




$$O\,R=O\,P+\tfrac{1}{2}(O\,S-O\,P)=\tfrac{1}{2}(O\,S+O\,P), \text{or } x=\tfrac{1}{2}(b+d)\cos\frac{A}{2}; \quad (1)$$

$$L\,R=M\,S-\tfrac{1}{2}(M\,S+P\,N)=\tfrac{1}{2}(M\,S-P\,N), \text{or } y=\tfrac{1}{2}(b-d)\sin\frac{A}{2}; \quad (2)$$

squaring (1) and (2) gives

$$\frac{x^2}{\cos^2\frac{A}{2}}=\tfrac{1}{4}(b+d)^2 \ (3), \text{ and } \frac{y^2}{\sin^2\frac{A}{2}}=\tfrac{1}{4}(b-d)^2 .. (4);$$

multiplying (3) and (4) by $(2-\cos A)$ and $(2+\cos A)$ respectively and adding, gives

$$x^2\frac{2-\cos A}{\cos^2\frac{A}{2}}+y^2\frac{2+\cos A}{\sin^2\frac{A}{2}}=\tfrac{1}{4}(b+d)^2(2-\cos A)+\tfrac{1}{4}(b-d)^2(2+\cos A)$$
$$=b^2+d^2-b\,d\cos A.$$

But by the conditions of the problem $b^2+c^2+d^2=8\,a^2$, also we have $c^2=b^2+d^2-2\,b\,d\cos A$; $\therefore\ b^2+d^2-b\,d\cos A=4\,a^2$, by eliminating $c^2$.

Hence
$$x^2\,\frac{2-\cos A}{\cos^2\frac{A}{2}}+y^2\,\frac{2+\cos A}{\sin^2\frac{A}{2}}=4\,a^2, \text{ or}$$

$$x^2\frac{2-\cos A}{1+\cos A}+y^2\,\frac{2+\cos A}{1-\cos A}=2\,a^2,$$

which is the equation of an ellipse referred to its axes.

If $A=60°$, $\cos A=\frac{1}{2}$, and $a=10$, the equation of the ellipse reduces to $x^2+5\,y^2=200$; if in this, $y=0$, $x=10\sqrt{2}$, if $x=0$, $y=2\sqrt{10}$; hence the values of the semi-axes are $10\sqrt{2}$ and $2\sqrt{10}$.

If $A = 90°$, we have $x^2 + y^2 = a^2$; hence if the given angle is a right angle, the locus is a circle whose centre is at the vertex of the right angle and radius $= a$.

This solution is by George A. Osborne, Jr.

Joseph Winlock.
Chauncey Wright.
Truman Henry Safford.

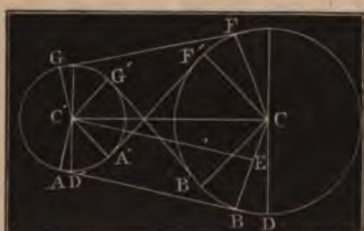
## SOLUTION OF PRIZE PROBLEM V., No. VII.

By J. B. Henck, C. E., Boston.

The problem of the belt and pulleys admits, perhaps, of a simpler solution, if we develop the arc and cosine in terms of the sine, instead of developing the sine and cosine in terms of the arc, as is done in the second solution, page 360. Another advantage is, that in this way a quite accurate approximate formula may be obtained, especially in the case of the open belt.

Let $B\,C = R$, $A\,C' = r$, $C\,C' = a$, and the length of the belt $= l$.

I. When the belt is open, denote the angle $A\,C'D' = B\,C\,D = C\,C'E$ by $\varphi$. Then the straight portions of the belt $= 2\,A\,B = 2\,C'E = 2\,a\cos\varphi$; also $R - r = C\,E = a\sin\varphi$. The length of the arc $B\,D\,F = R(\pi + 2\varphi) = R\,\pi + 2\,R\,\varphi$; that of the arc $A\,G = r\,(\pi - 2\,\varphi) = (R - a\sin\varphi)\,(\pi - 2\,\varphi) = R\,\pi - 2\,R\,\varphi - a\,\pi\sin\varphi + 2\,a\,\varphi\sin\varphi$. Adding the straight and curved portions, we have



$$2\,a\cos\varphi + 2\,R\,\pi - a\,\pi\sin\varphi + 2\,a\,\varphi\sin\varphi = l;$$

$$(1) \qquad \therefore\ 2\cos\varphi - \pi\sin\varphi + 2\,\varphi\sin\varphi = \frac{l - 2\,R\,\pi}{a}.$$

Let $\sin\varphi = x$. Then $\varphi = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 +$ &c., and $\cos\varphi = \sqrt{1-\sin^2\varphi}$ $= \sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 -$ &c. Substituting these values of $\varphi$ and $\cos\varphi$ in (1), we have

$$2\left(1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \text{\&c.}\right) - \pi x + 2x\left(x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \text{\&c.}\right) = \frac{l - 2R\pi}{a}.$$

$$(2) \qquad \therefore x^2 - \pi x + \tfrac{1}{12}x^4 + \tfrac{1}{40}x^6 + \text{\&c.} = \frac{l - 2R\pi}{a} - 2.$$

The value of $x$ being found from (2), we have $r = R - a\sin\varphi$ $= R - ax$.

Now since $x$ is a proper fraction, generally small, we may obtain an approximate formula by neglecting $\frac{1}{12}x^4 + \frac{1}{40}x^6 +$ &c.

Hence,

$$x^2 - \pi x = \frac{l - 2R\pi}{a} - 2;$$

$$\therefore x = \tfrac{1}{2}\pi \pm \sqrt{\tfrac{1}{4}\pi^2 + \frac{l - 2R\pi}{a} - 2};$$

$$(3) \qquad \therefore x = 1.5708 \pm \sqrt{.46740110 + \frac{l - 6.28318531 R}{a}}.$$

II. When the belt is crossed, denote the angle $A'C'D' = B'CD$ $= 90° - B'CC'$ by $\varphi$. Then if a perpendicular be dropped from $C'$ on $CB'$ produced, it will be seen that each of the straight parts of the belt $= A'F' = B'G' = CC'\cos\varphi = a\cos\varphi$; also that $R + r = a\sin\varphi$. The length of the arc $B'DF' = R(\pi + 2\varphi)$; that of the arc $A'D'G'$ $= r(\pi + 2\varphi)$; so that the sum of the curved portions $= (R + r)$ $(\pi + 2\varphi) = a\sin\varphi\,(\pi + 2\varphi)$.

$$\therefore 2a\cos\varphi + a\pi\sin\varphi + 2a\varphi\sin\varphi = l;$$

$$\therefore 2\cos\varphi + \pi\sin\varphi + 2\varphi\sin\varphi = \frac{l}{a}.$$

Substituting the same values of $\varphi$ and $\cos\varphi$ as above, we have

$$(4) \qquad x^2 + \pi x + \tfrac{1}{12}x^4 + \tfrac{1}{40}x^6 + \text{\&c.} = \frac{l}{a} - 2.$$

The value of $x$ being found from (4), we have $r = a \sin\varphi - R = a\,x - R$.

The approximate formula is

$$x^2 + \pi x = \frac{l}{a} - 2;$$

$$\therefore\ x = -\tfrac{1}{2}\pi \pm \sqrt{\tfrac{1}{4}\ {}^2 + \frac{l}{a} - 2};$$

$$(5) \qquad \therefore\ x = -1.5708 \pm \sqrt{.46740110 + \frac{l}{a}}.$$

The formula in this case is not so accurate as in the first case, because $x$ is generally a larger fraction. In both cases a second approximation is very readily made by subtracting from the quantity under the radical in (3) and (5) the value of as many terms of $\frac{1}{12}x^4 + \frac{1}{40}x^6 +$ &c., as may be thought necessary, computed from the first approximate value of $x$. A third approximation may be made in a similar way.

This method of approximation, it is evident, may be applied to finding the incommensurable roots of numerical equations of any degree; since it is easy to transform such an equation so that its roots shall be proper fractions. The method occurred to me several years ago, while solving this same problem. It is more than probable, however, that so obvious a process has been tried before, and found more laborious than the ordinary ones.

## NOTES AND QUERIES.

[Continued from page 264.]

8. THE following tabular summary of the notation and definitions of algebra may be used with advantage in teaching beginners. Write the table on the blackboard, or on a large card, and let the class review it daily until it is fixed in the mind. Let $a, b, c,$ &c.

denote any numbers whatever; their relations and the corresponding symbols have the following names:—

| Symbol. | Name of Symbol. | Relation. | Name of Relation. | |
|---|---|---|---|---|
| $+$ | *Plus.* | $a+b$ | *Sum*: | $b$ added to $a$. |
| $-$ | *Minus.* | $a-b$ | *Diff.*: | $b$ subtracted from $a$. |
| $\times$, ., nothing. | *Multiplied by.* | $a\times b, a.b, ab$ | *Product*: | $a$ multiplied by $b$. |
| $\div$ fraction. | *Divided by.* | $a\div b, \frac{a}{b}$ | *Quotient*: | $a$ divided by $b$. |
| $=$ | *Equals.* | $a+b=c-d$ | *Equation*: | sum equals diff. |
| $<$ | *Less than.* | $a<b$ | *Inequality*: | $a$ less than $b$. |
| $>$ | *Greater than* | $a>b$ | *Inequality*: | $a$ greater than $b$. |
| $:$ | *Ratio.* | $a:b$ | *Quotient*: | $a$ divided by $b$. |
| $::$ | *Equality of ratios.* | $a:b::c:d$ | *Proportion*: | equality of ratios. |
| —— | *Vinculum or bar.* | $\overline{a+b}\times c, \frac{a+b}{c}$ | Sum multiplied or divided by $c$. | |
| ( ) | *Parenthesis.* | $a-(b-c+d)$ | $(b-c+d)$ as a whole, subtr'd from $a$. | |
| $(\ )^n$ | *Exponent.* | $(a+b)^n, c^n$ | *Power*: $a+b$ or $c$, taken $n$ times as a factor. | |
| $\sqrt[n]{\ }, (\ )^{\frac{1}{n}}$ | *Root-sign.* | $\sqrt[n]{a+b}, (a+b)^{\frac{1}{n}}$ | *Root*: $n$th root of $a+b$. | |
| $'\ ''\ '''$ &c. | *Accents.* | $a', a'', a'''$ &c. | Symmetry of notation. | |

This idea each teacher can extend and modify at pleasure.

*Remark* 1. These symbols, or many of them, are usually defined in the arithmetics, but hardly ever used. Why is this? How easy it would be to make pupils entirely familiar with them in connection with *numbers*, about which their ideas are clear and fixed. All these symbols ought constantly to be used with small numbers, until they are indelibly associated in the mind with the processes they indicate; and all operations ought to be indicated before they are performed. Thus,

$$2\sqrt{\frac{\overline{(2+5-3)}\times\overline{8+12}}{11}}=2\sqrt{\frac{4\times\overline{8+12}}{11}}=2\sqrt{\tfrac{44}{11}}=2\sqrt{4}=4.$$

If this were done, how easy and obvious the transition to algebra would be. Arithmetic, and the simple elements of algebra, are very intimately connected; and by thus gradually passing from one to the other, we shall make the notation, which has been too much

considered peculiar to algebra, useful in explaining the operations of arithmetic; while, on the other hand, these arithmetical operations furnish the simplest practical illustrations of the results of algebra.

After the pupil is once familiar with all the symbols, how easy it would be to introduce into numerical problems the answer as a quantity, not perhaps at first denoting it by a letter, but by some significant word.

*Remark* 2. It is usually much better to write a proportion in the form $\frac{a}{b}=\frac{c}{d}$, than $a:b::c:d$. Because, in all the transformations,

$$\frac{a}{c}=\frac{b}{d}\,,\ \frac{d}{b}=\frac{c}{a}\,,\ \frac{a}{a+b}=\frac{c}{c+d}\,,\ \frac{a}{c}=\frac{a+b}{c+d}\,,\ \frac{a-b}{a+b}=\frac{c-d}{c+d}\,,\ \text{\&c.,}$$

the symmetry, which is more directly apparent, is of great service.

9. *Simplification of the Expression* $\sqrt{a \pm c\sqrt{d}}$. The remarks of Mr. John M. Richardson, in No. II., on the reduction of expressions of this form, have induced me to refer the readers of the Monthly to Benedict's Algebra, page 152, where the subject is treated in the same manner. We will add a single example, which may be taken as the representative of others, that at first seems to be incapable of reduction in the manner proposed. Simplify the expression $\sqrt{2x \pm 2\sqrt{x^2-1}}$. Here the correspondence between the rational and radical parts of the expression may be seen as follows.

The rational part $2x=(x+1)+(x-1)$.

The radical part $2\sqrt{x^2-1} = 2\sqrt{(x+1)(x-1)}$.

$$\therefore \sqrt{2x \pm 2\sqrt{x^2-1}} = \sqrt{x+1 \pm 2\sqrt{(x+1)(x-1)}+x-1} = \sqrt{(\sqrt{x+1} \pm \sqrt{x-1})^2}$$
$$= \sqrt{x+1} \pm \sqrt{x-1}.$$

B.

10. If $R$ be the radius of a sphere, prove that the edges of the five regular solids circumscribing and inscribing it are respectively as follows; namely, for the tetraedron, $2R\sqrt{6}$ and $\frac{2}{3}R\sqrt{6}$; for the

hexaedron, $2R$ and $\frac{2}{3}R\sqrt{3}$; for the octaedron, $R\sqrt{6}$ and $R\sqrt{2}$; for the dodecaedron, $R\sqrt{50-22\sqrt{5}}$ and $\frac{1}{3}R(\sqrt{15}-\sqrt{3})$; for the icosaedron, $R\sqrt{3}(3-\sqrt{5})$ and $\frac{1}{5}R\sqrt{10(5-\sqrt{5})}$. Joseph Ficklin, Jr., Trenton, Missouri.

11. "If the digits of a given number be transposed in any order, the difference between the number thus formed and the given one is divisible by 9."

Any digit $x$ of the given number will have some positive integer power, $m$, of 10, including zero, for its coefficient; and in the new number, if $x$ has changed its place, it will have some other power, $n$, of 10 for its coefficient. Therefore, in the difference, the coefficient of $x$ will be $10^m-10^n$; and since the coefficients of all the other digits will have the same form in the difference, it will be sufficient to prove that $10^m-10^n$ is divisible by 9. But $10^m-10^n=(9+1)^m-(9+1)^n$, and, by expanding these binomials, the last terms of these expansions being unity with opposite signs, will cancel each other, while all the other terms will contain 9 as a factor; therefore $10^m-10^n$ is divisible by 9, and the theorem is proved.

Cor. By the same reasoning it appears that in a system of numeration of which the base is $b$, the highest digit, $(b-1)$, would be invested with the same property as 9 in our system. John F. Lanneau, Furman University, Greenville, South Carolina.

12. *Note on Logarithms.* A correspondent invites attention to the convenience of distinguishing between the direct and inverse use of logarithmic tables. When a number is given to find its logarithm, the sign of equality, or a blank space is commonly employed; thus, $l\,436.2=2.63969$; $l\,436.2\quad 2.63969$; $l\,436.2\ldots.2.63969$. In the last case, the three or four dots imply that the number was found *from the logarithm.* When such a number or its logarithm is to be again employed in a case where greater accuracy is required, the

dots will show when the last decimals are to be reëxamined and corrected before making such application.

Since the prefix *log* is employed in analysis to denote Napierian or hyperbolic logarithms, the next most convenient designation which remains, is the italic *l* for designating common logarithms exclusively.

## NOTE ON DIFFERENTIATION.

By Simon Newcomb, Naut. Alm. Office, Cambridge, Mass.

The following is an example of a method of finding a derivative by means of some special property of the function.

To find the derivative of a function which satisfies the condition

$$\psi\,(x+y)=\psi\,x\,.\,\psi\,y.$$

Representing the successive derivatives by $\psi'$, $\psi''$, &c., we have

$$\psi'\,(x+y)=\psi\,x\,.\,\psi'\,y=\psi'\,x\,.\,\psi\,y.$$

That this equation may be satisfied, it is necessary and sufficient that we have

$$\psi'\,x=m\,\psi\,x,$$

$m$ being a constant to be determined. By successive differentiation and substitution

$$\begin{aligned}\psi''\,x&=m\,\psi'\,x=m^2\,\psi\,x\\ \psi'''\,x&=m\,\psi''\,x=m^3\,\psi\,x\\ &\vdots\qquad\qquad\vdots\qquad\quad \&c.\end{aligned}$$

Whence, by Maclaurin's theorem

$$\psi\,x=\psi\,0\left(1+mx+\frac{m^2x^2}{2\,!}+\frac{m^3x^3}{3\,!}+\&c.\right).$$

Put $x=\frac{1}{m}$, and we have

$$\psi\,\frac{1}{m}=2.71828\ldots.\,(\psi\,0)\text{ which determines }m.$$

By the above method the function $a^x$ may easily be differentiated. We have $D_x a^x = m a^x$; where $a^{\frac{1}{m}} = 2.71828....$ $\therefore (2.71\!\!> ..)^m = a$ or $m$ is $\log a$ in the system of which the base is $2.7182818....$

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THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 373.]

43. The pressure of the atmosphere may be obtained from the first of equations (9). The terms in the equation depending upon the motions of the atmosphere are insensible, and consequently may be omitted. The term $D_t^2 r$ depends upon the acceleration or retardation of the vertical motion of the atmosphere, and is of the same order in comparison with $g$ as the rate of its acceleration or retardation in comparison with that of a descending or ascending free body, and hence in all ordinary motions of the atmosphere it is insensible. Restoring the value of $\omega$ in (§ 2), the largest of the remaining terms is $2 r \sin^2 \theta \, n \, D_t \varphi$, which is of the same order in comparison with $r n^2$, as the east or west motion of the atmosphere in comparison with the motion of the rotation of the earth on its axis. But $r n^2 = \frac{1}{289} g$ only, hence the term is entirely insensible. We may therefore put

$$\frac{D_N P}{P} = - \alpha g.$$

Using the common system of logarithms and putting $M$ for its modulus, we get by integration

$$(54) \qquad \log P' - \log P = M \alpha g N,$$

in which $P'$ is the pressure at the earth's surface.

Hence

$$(55) \qquad D_{\theta'} \log P = D_{\theta'} \log P' - M g N D_{\theta'} \alpha.$$

By means of this equation the second of equations (9) becomes, by putting $\alpha P$ for $k$, (§ 6), and omitting the very small term $2 r D_t r D_t \theta$,

(56) $D_{\theta'} \log P' - M g N D_{\theta'} \alpha = M \alpha \left[ r^2 \sin\theta \cos\theta \, (2 n + D_t \varphi) \, D_t \varphi - r^2 D_t^2 \theta \right].$

44. In the case of the atmosphere, there must be a term in equations (9) to represent the resistances to the motions; and this term in the second of these equations may be denoted by $(\varphi) D_t \theta$ in the second member. Putting

(57) $$W = (\varphi) D_t \theta + r D_t^2 \theta,$$

$W$ will represent the force which overcomes the resistances to the motions of the atmosphere between the equator and the poles, and also its inertia.

Since $D_t \varphi$ is generally very small in comparison with $2 n$, it may be omitted in equation (56), which then becomes, by means of equation (57),

(58) $D_{\theta'} \log P' - M g N D_{\theta'} \alpha = M \alpha (2 r^2 n \sin\theta \cos\theta \, D_t \varphi - r W).$

45. It will be shown, that $r W$ is very small in comparison with $2 r^2 n \sin\theta \cos\theta \, D_t \varphi$; and hence, if $D_t \varphi$ were known, the preceding equation, neglecting the term $r W$, would give approximately the pressure of the atmosphere at the earth's surface. But since we do not know the value of the term denoting the resistances in the last of equations (9), we cannot determine the value of $D_t \varphi$; therefore, since $D_{\theta'} \log P'$ can be determined from observations of the barometric pressure, we shall use the equation to determine $D_t \varphi$, from which we easily obtain the east or west motion of the atmosphere. Denoting the velocity of this motion per hour by $v$, we shall have

(59) $$v = 3600 \, r \sin\theta \, D_t \varphi.$$

46. The ratio of the density to the elastic force decreases $\frac{1}{491}$ for

every degree of Fahrenheit. But as a higher temperature is always accompanied by a greater amount of aqueous vapor, the density of which is less than that of the atmosphere, the rate of decrease has been found to be $\frac{1}{449}$ for every degree. Let

$\alpha'$ be the value of $\alpha$ at the equator, and

$i$ the difference of temperature between the equator and the poles.

If we suppose the temperature to decrease from the equator to the poles as the square of the sine of the latitudes, we shall have

$$\alpha = \alpha' (1 + \tfrac{1}{449}\, i \cos^2 \theta).$$

Hence

$$D'_\theta\, \alpha = - \tfrac{2}{449}\, \alpha'\, i \sin \theta \cos \theta.$$

By means of the last three equations, equation (58), putting $R$ for $r$ and $e$ for $\frac{1}{M \alpha' g}$, is reduced to

$$(60)\quad v = \frac{1800}{R\, n \cos \theta\, (1 + \frac{1}{449}\, i \cos^2 \theta)} \Big( e\, g\, D_{\theta'} \log P' + \tfrac{2}{449}\, i\, g \sin \theta\, N + W \Big).$$

Since the variation of $\alpha$ with the altitude can produce no sensible effect in the results, $\alpha$ has been regarded as a function of the latitude only. We must, therefore, take the mean value of $\alpha'$ belonging to the atmosphere at the equator at all altitudes, which we will assume to be that belonging to the temperature of 32°.

47. By means of observations of $P$ at different altitudes, equation (54) gives the value of $\frac{1}{M \alpha g}$, which, at the temperature of 32°, has been determined to be 60156 feet; which, consequently, is the value of $e$. The difference between the mean temperatures of the equator and the poles is about 60°; we shall, therefore, in the following applications, put $i = 60$.

48. The value of $D_{\theta'} P'$ in the preceding equation can be determined approximately for any latitude from the preceding tables of barometric pressure. Since the coördinates of pressure given there

have been deduced from observations made in different longitudes and at all seasons, they are somewhat irregular; but coördinates can be assumed with regular differences, and such that the interpolated values of the coördinates of pressure for the latitudes given in the tables will very nearly correspond with the pressures given there; and then, from these coördinates, the approximate value of $D_{\theta'} \log P'$ can be determined. In this manner the values of $D_{\theta'} P'$ in the following table have been determined, except the first, which has been assumed. The third column of the table contains the values of $v$ at the earth's surface, neglecting the term $W$, which will be shown to have, in general, a very little effect. The fourth contains the coefficient of $N$, and the fifth the value of $v$ at the height of 3 miles.

TABLE E.

| Latitude. | $D_{\theta'} \log P'$. | $v$, ($N=0$). | Coeff. of $N$. | $v$, ($N=3$ miles). |
|---|---|---|---|---|
| 75 N. | —.0060 | — 2.7 miles. | 2.33 | + 4.3 miles. |
| 65 | .0000 | 0.0 | 3.87 | 11.6 |
| 55 | +.0188 | + 9.9 | 5.34 | 25.9 |
| 45 | +.0080 | + 4.5 | 6.71 | 24.6 |
| 30 | .0000 | 0.0 | 8.49 | 25.5 |
| 15 | —.0060 | —10.0 | 9.70 | 19.1 |
| 15 S. | +.0060 | —10.0 | 9.70 | 19.1 |
| 30 | —.0147 | +11.4 | 8.49 | 36.9 |
| 40 | —.0372 | +23.4 | 7.36 | 45.5 |
| 50 | —.0295 | +15.3 | 6.07 | 33.5 |
| 60 | —.0133 | + 6.0 | 4.61 | 19.8 |

49. The term $W$, and its effect upon the value of $v$, cannot be determined, but they can be shown from observation to be, in general, very small; and, since $W$ is positive, as may be seen from equation (75), when the motion is from the north towards the south, and negative when the contrary, except when the motion is retarded, and the term $r\,D_t^2\,\theta$ arising from the inertia of the atmosphere is greater

than the resistances, its effect for the most part is to increase the value of $v$ algebraically where the motion is towards the south, and decrease it where it is towards the north. In the regions of the trade winds about the parallels of 15°, the current at the surface of the earth is stronger than at any other parallel, and as the resistances at the surface must be much greater than in the upper regions, the term $W$ must be greater there than in any other part of the atmosphere. If $v = 0$, equation (60) gives, since $N = 0$ at the surface, $W = e g D_{\theta'} \log P'$; and from the preceding table, when $v = -10$ miles, $W = 0$. Now, we know from observation that the velocity of the atmosphere westward at the parallels of 15°, cannot be much less than 10 miles per hour, and hence $W$ is small in comparison with $e g D_{\theta'} \log P'$, which at that parallel is itself small; and hence the effect of $W$ upon the value of $v$ in the higher latitudes, where the value of $\cos \theta$ in the denominator is much greater, must be very small. Very near the equator the formula for the value of $v$, equation (60), fails practically, since, on account of the small value of $\cos \theta$ there, the effect of $W$ may be very great.

50. If the motion of the atmosphere east in the higher latitudes and west near the equator, be that given in the preceding table, or by equation (60), it must cause the observed difference of barometric pressure in the different latitudes; and hence, from what we know of those motions of the atmosphere from observations, there can be no doubt that they are adequate to account for this observed difference of pressure.

51. It is evident, where the motions of the atmosphere are resisted by the earth's surface, that all the conditions cannot be satisfied by a motion at the surface from the poles towards the equator, and by a counter motion in the upper regions. For we have seen (§ 35), that the atmosphere at the surface of the earth must have an eastern motion in the middle latitudes; but it cannot have such a motion,

unless it also have a motion towards the poles, in order that the deflecting force (§ 18) arising from this motion may overcome the resistances to the eastern motion. But it is evident that there cannot be a complete reversal of the motions in the middle latitudes, but some portion of it must flow towards the poles in the upper regions, else the eastern motion there could not be greater than at the surface, which the conditions require. The motions, therefore, must be somewhat as represented in Fig. (5). The part of the atmosphere next the earth's surface in the middle latitudes having a motion towards the poles, extends to a considerable height, since it generally embraces the region of fair weather clouds, as may be seen by observation.

52. It is seen, from the results given in the preceding table, that the eastward motion of the atmosphere in the middle and higher latitudes must be greatest in the upper strata, and that in the region of the trade winds, where the motion is westward at the surface, it must be towards the east above. This is also evident from the general consideration, that the whole amount of deflecting force eastward arising from the motion of the atmosphere towards the poles is equal to the deflecting force westward arising from its motion back towards the equator, and that the deflecting force eastward is principally above where there is less resistance than near the surface. Hence at the top of Mauna Loa in the Sandwich Islands, and on the peak of Teneriffe, both of which places are near the tropical calm belt at the surface, a strong south-west wind prevails. Hence, also, "on the eruption of St. Vincent, in 1812, ashes were deposited at Barbadoes, sixty or seventy miles eastward, and also on the decks of vessels one hundred miles still further east, whilst the trade wind at the surface was blowing in its usual direction." The eastward motion of the atmosphere above, in the latitudes of the trade winds, is also confirmed by observations made on the directions of the

clouds at Colonia Tovar, Venezuela, latitude 10° 26′, as given in the Report of the Smithsonian Institution for 1857 (p. 254). While the motion of the lower clouds was in general from some point towards the east, the observed motion of nearly all the higher clouds was from some point towards the west.

53. From what precedes, the limit between the atmosphere which moves eastward in the middle latitudes and westward nearer the equator, which at the earth's surface is at the tropical calm belt, must be a plane inclining towards the equator above. And since, according to (§ 51), the atmosphere near the earth's surface cannot have an eastward motion, unless it also has a motion toward the poles; this plane near the earth's surface must nearly coincide with the one which separates the atmosphere moving towards the poles from that moving towards the equator, in the trade wind regions, and hence the latter must also incline above towards the equator. This explains the winds at the peak of Teneriffe, which at the top always blow from the south-west, while at the base they blow alternately from the south-west and north-east, changing with the seasons. As the tropical calm belt together with this dividing plane changes its position with the seasons, as will be explained, in the latter part of summer when this plane is farthest north, it still leaves the top of the peak north of it while the base is south of it; and hence the wind at the top always blows from the south-west, even when at the base it blows from the north-east. As this plane moves south in the fall, more of the peak gradually becomes north of it; and hence the south-west wind, which always prevails at the top, gradually descends lower on the sides of the peak until it reaches the base. Hence, when this plane reaches its most southern position, in the latter part of winter, the south-west wind prevails at both the base and the top.

54. It is seen, from the first of the results given in the last table,

that if the barometric pressure increases near the poles, as it seems to do, at least in the northern hemisphere, the atmosphere at the earth's surface must have a westward motion there; and as it cannot have this motion unless it also have a motion toward the equator, so that the deflecting force arising from this motion may overcome the resistance to the westward motion, the wind there must blow slightly from the north-east, as has been shown in (§ 37). This, according to Professor COFFIN's chart of the winds, already alluded to, seems to accord with observation.

55. The depression of the atmosphere at the poles and at the equator, and the accumulation near the tropics, may be explained in a general manner by means of the principle in (§ 32), that when a body moves in any direction in the northern hemisphere, it is deflected to the right, and the contrary in the southern. The atmosphere towards the poles having an eastward motion, the deflecting force arising from it causes a pressure towards the equator, and the motion near the equator being westward, the pressure is towards the poles; and hence there must be a depression at the poles and at the equator, and an accumulation near the tropics. Since this deflecting force is as $\cos\theta$, it is small near the equator; and, consequently the depression there is small.

56. According to the preceding tables of barometric pressure, there is more atmosphere in the northern than in the southern hemisphere. Says Sir JAMES ROSS, "the cause of the atmosphere being so very much less in the southern than in the northern hemisphere remains to be determined." This is very satisfactorily accounted for by the preceding principle; for as there is much more land, with high mountain ranges, in the northern hemisphere, than in the southern, the resistances are greater, and consequently the eastward motions, upon which the deflecting force depends, is much less; and the consequence is, that the more rapid motions of the

southern hemisphere cause a greater depression there, and a greater part of the atmosphere to be thrown into the northern hemisphere.

This also accounts for the mean position of the equatorial calm belt being, in general, a little north of the equator. But in the Pacific Ocean, where there is nearly as much water north of the equator as south, its position nearly coincides with the equator.

For the same reason the tropical calm belt of the northern hemisphere is farther from the equator than that of the southern hemisphere; and, on account of the irregular distribution of the land and water of the two hemispheres in different longitudes, it does not coincide with any parallel of latitude. In the longitude of Asia, where there is all land in the northern hemisphere and the Indian Ocean in the southern, this belt, which is also the dividing line which separates the winds which blow east from those which blow west, is farther from the equator than at any other place, as shown by Professor Coffin's chart.

57. In winter, the difference of temperature between the equator and the poles, upon which the disturbance of the atmosphere depends, is much greater than in summer; this causes the eastward motion of the atmosphere in either hemisphere during its winter to be greater, while in the other hemisphere it is less. Hence a portion of the volume of the atmosphere in winter is thrown into the other hemisphere; but, although the volume or height of the atmosphere is then less, yet, being more dense, the barometric pressure remains nearly the same. The difference at Paris, and in the middle latitudes generally, between winter and summer, is only about $\frac{1}{10}$ of an inch.

On account of this alternate change with the seasons of the velocity of the eastward motion of the atmosphere in the two hemispheres, the equatorial and tropical calm belts change their positions

a little, moving north during our spring, and south in the fall. When the sun is near the tropics, the true law of the decrease of temperature from the equator to the poles varies from that which has been assumed, (§ 46), and is then different in the two hemispheres, which doubtless has some effect also upon the position of the calm belts.

[To be Continued.]

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## METHOD OF SOLVING NUMERICAL EQUATIONS.

BY DASCOM GREENE,
Professor of Mathematics in Rensselaer Polytechnic Institute, Troy, N. Y.

THE most direct method of resolving a numerical equation of any degree, is to transform it in such a way that its first member becomes a perfect power of the same degree; the solution is then reduced to extracting the root of its second member.

If the first member can be made a perfect *n*th power by the addition or subtraction of a *number*, the solution will be effected by simply extracting the *n*th root of the absolute term.

Thus, if the given equation be

$$x^3 - 3x^2 + 3x = 15,$$

subtract . . . . . . . . $1 = 1;$

then $(x-1)^3 = x^3 - 3x^2 + 3x - 1 = 14,$

$\therefore$ $x - 1 = \sqrt[3]{14}$, and $x = 1 + \sqrt[3]{14}$.

If, however, it should be necessary to add terms involving $x$, then the quantity whose *n*th root is required, will be a polynomial, with numerical coefficients, and of a degree inferior to the *n*th. Thus, if we have

$$x^3 + 7x^2 + 5x = 18,$$

add $8 - x^2 + 7x = 8 - x^2 + 7x;$

then $\quad (x+2)^3 = x^3 + 6x^2 + 12x + 8 = 26 - x^2 + 7x,$

$$\therefore \qquad x = -2 + \sqrt[3]{26 - x^2 + 7x}.$$

We proceed, as before, to extract the root of the absolute term, but take care, during the process, to correct each partial dividend for the value of $x$ already found.

EXAMPLE 1. Find a root of the equation

$$x^3 + 2x^2 + 3x = 13089030.$$

Add $\qquad x^2 + 1 = x^2 + 1;$

then $\qquad x^3 + 3x^2 + 3x + 1 = 13089031 + x^2,$

and $\qquad x + 1 = \sqrt[3]{13089031 + x^2}.$

Hence, in proceeding to extract the cube root of the absolute term, each successive dividend must be increased by the value of $x^2$ corresponding to that part of the root already found. We have the following operation*:

$$\begin{array}{rrlrr}
 & 13089031 & \mid \underline{236} = x+1 \quad \therefore x = 235. & & \\
\text{If} \qquad x = 200,\ x^2 = & \underline{40000} & & & \\
 & 13129031 & & & \\
 & \underline{8\phantom{000000}} & = 2^3 & & \\
 & 5129031 & 3 \times 2^2 & = 12.. & \mid 12.. \\
230^2 - 200^2 = 30 \times 430 = & 12900 & 3 \times 2 \times 3 & = 18. & \mid 36. \\
 & 5141931 & & 3^2 = 9 & \mid 27 \\
 & \underline{4167\phantom{000}} & = 3 \times & 1389 & \mid 1587.. \\
 & 974931 & & 3 \times 23 \times 6 = & 414. \\
235^2 - 230^2 & \underline{= 2325} & & 6^2 = & \underline{36} \\
 & 977256 & & & \\
 & 977256 & & & = 6 \times 162876
\end{array}$$

* We employ here the method of extracting roots explained in the January number of the MONTHLY, page 123, and which is originally due to Prof. GEORGE C. WHITLOCK, of Coburg, Canada West.

EXAMPLE 2. Find a root of the equation

$$x^4 - 3x^2 + 75x = 10000.$$

We have $x = \sqrt[4]{10000 + 3x^2 - 75x}$.

| | | | | |
|---|---|---|---|---|
| | 10000 | $\vert$ 9.886 .... $= x$ | | |
| $3 \times 9^2 - 75 \times 9 =$ | 432 | | | |
| | 9568 | | | |
| $9^4 =$ | 6561 | $4 \times 9^3 = 2916...$ | 2916... | |
| | 3007.0000 | $6 \times 9^2 \times 8 = 3888..$ | 7776.. | |
| $\left.\begin{matrix}3(9.8^2 - 9^2)\\ -75 \times 0.8\end{matrix}\right\}$ $=$ | 14.88 | $4 \times 9 \times 8^2 = 2304.$ | 6912. | |
| | 2992.1200 | $8^3 = 512$ | 2048 | |
| | 2662.6816 | $= 0.8 \times 3328.352$ | 3764768.... | 3764768... |
| | 329.43840000 | $6 \times 98^2 \times 8 = 460992..$ | | 921984.. |
| $\left.\begin{matrix}3(9.88^2 - 9.8^2)\\ -75 \times 0.08\end{matrix}\right\}$ $=$ | $-1.2768$ | $4 \times 98 \times 8^2 = 25088.$ | | 75264. |
| | 328.16160000 | $8^3 = 512$ | | 2048 |
| | 304.88948736 | $= 0.08 \times 3811.118592$ | | 3857721088 |
| | 23.272112640000 | | $6 \times 988^2 \times 6 = 35141184$ | |
| $\left.\begin{matrix}3(9.886^2 - 9.88^2)\\ -75 \times 0.006\end{matrix}\right\}$ $=$ | $-0.094212$ | | $4 \times 988 \times 6^2 = 142272$ | |
| | 23.177900640000 | | $6^3 = 216$ | |
| | 23.167419776016 | | $= 0.006 \times 3861.236629336$ | |

This method of solving equations is of universal application, but its facility depends entirely on the skill of the computer, in making the necessary transformations. It is due, so far as I am aware, to Mr. JACOB HAFF, of Plumb Brook, Michigan.

# A SECOND BOOK IN GEOMETRY.

## REASONING UPON FACTS.

BY THOMAS HILL.

Continued from page 285.]

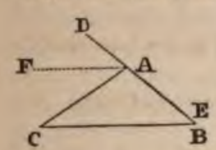
## CHAPTER V.

### VARIETY OF PATHS.

54. As there are usually many paths by which we may ascend a hill, so there are usually many modes by which we may demonstrate a proposition. In the case of a simple proposition, it is not usually worth while to try more than one mode. But with more difficult problems, it is sometimes worth while to spend a great deal of labor in discovering the simplest mode of demonstration. There are geometrical truths which can be demonstrated in so simple a manner as to require only twenty lines to write down the demonstration; and yet some writers, from ignorance of this simple mode, have written more than twenty pages to prove the same truths.

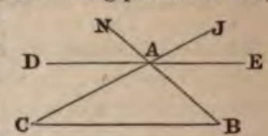
55. It will therefore be useful to you, to show you, by a simple example, such as that of the equality of the sum of the angles in a triangle to two right angles, the great variety of methods by which a single proposition can be proved.

56. In the proofs of this proposition, which I will now give you, I will not be careful to follow out every step. It will be enough, *for the purpose we now have in view*, simply to show you the general line of the paths, without taking you through every step of the way.



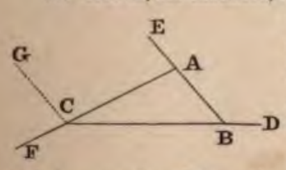

57. The line D E might be drawn so as to coincide in direction with one of the other sides of the triangle, as A B, which would give us the figure in the margin. And by imagining the dotted line A F parallel to C B, we should have F A C equal to A C B, and F A D equal to C B A, which would make the three angles at A equal to the three angles of the triangle, as in the former proof. (See chapter III.)

58. By prolonging the lines C A and B A through the point A, D E being parallel to B C, we should have N A D equal to C A B; J A E equal to A C B, and N A J equal to C A B. So that the three angles of the triangle will be equal to the three angles on the upper side of the line D E, which are manifestly equal to two right angles.



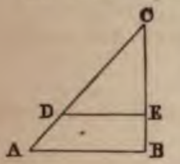

59. The three methods of proving this proposition that I have now given, are strictly geometrical. Others might be given, that are something more like algebraic reasoning.

60. Let us, for instance, imagine each side of a triangle prolonged at its right hand end, as in this figure, and also a line drawn from one vertex, as C, parallel to the opposite side B A. Now the external angles F C B, D B A, E A F, are plainly equal to the three angles F C B, B C G, G C F; and these amount to four right angles. But each external angle is plainly the supplement of one of the angles of the triangle; that is, it is equal to the difference between two right angles and one angle of the triangle. The sum of the three external angles must therefore be



equal to the difference between six right angles and the angles of the triangle. But as this difference is four right angles, the three angles of the triangle must be equivalent to two right angles.

61. If we introduce the idea of motion, we can devise quite a different sort of demonstration.




Suppose, for instance, that I stand at the point A, with my face towards C. Let me now turn to the right until I face towards B. I have now changed the direction of my face by an amount which is equal to the angle at A. Suppose that I now walk to B, without turning; I shall have my back towards A, and if standing still, I turn to the right, until I have changed my direction by an amount equal to the angle A B C; I shall have my back towards C. Let me now walk backward without turning, until I reach C, and I shall have my face towards B. I will now turn a third time to the right, until I face the point A. My three turnings, or changes of direction, have been equal to the three angles of the triangle; they have all been to the right; therefore my whole change of direction is equal to the sum of these angles; I am now looking in exactly the opposite direction to that from which I started; I am looking from C to A, instead of from A to C; I have turned half way round; that is, through two right angles. Whence, the sum of the three angles of the triangle is equivalent to two right angles.

62. Another demonstration, by means of motion, may be obtained as follows. Suppose an arrow, longer than either side of the triangle, to be laid upon the side A C, pointing in the direction from A to C. Taking hold of the pointed end beyond C, turn the arrow round upon the point A, as a pivot, until the arrow lies upon the line A B. Taking now hold of the further end, beyond A, turn the arrow upon B as a pivot, until the arrow lies upon the line B C. Using C as a pivot, turn it now until the point of the arrow is over A. The arrow has thus been reversed in direction, turned half way round, or through two right angles. It has been turned successively through the three angles of a triangle, and every time in the same direction, like the hands of a watch; so that its total change of direction, two right angles, is equivalent to the sum of the three angles.

63. You have thus seen how a single proposition may be proved in a variety of ways. We have shown what is the value of the sum of the angles in a triangle, in six different ways; in three, by what is called rigid geometry, in one by a partly algebraical process; and in two, by introducing the idea of motion. And I wish you to observe, that every one of the six ways is satisfactory. They are all proofs that are certain; because they lead you from self-evident truths, by self-evident steps. One is not more certain than the other, because they are all absolutely certain. The only choice between them is, that some are more purely geometrical; some are better adapted to the peculiar tastes of different students, and some are neater and more quickly perceived by untaught persons.

# Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the June number of the MONTHLY: —

JOHN W. JENKS, Senior Class, Columbia College, N. Y., answered questions III., IV., and V. (W. G. PECK, Prof.)

GEORGE A. OSBORNE, Jr., Lawrence Scientific School, answered all the questions. (H. L. EUSTIS, Prof.)

JAMES D. WARNER, West Troy, N. Y., answered all the questions but Problem IV.

ASHER B. EVANS, Madison University, Hamilton, N. Y., answered all the questions. (L. M. OSBORNE, Prof.)

D. D. ROSA, Georgetown, South Carolina, answered questions I. and II.

WILLIAM B. CLEVELAND, Lawrence Scientific School, answered all the questions. (H. L. EUSTIS, Prof.)

WE have, in one or two cases, unintentionally omitted to give credit for Solutions, which, we regret to say, have been mislaid. . . . . . The Report of the Judges on the Prize Essays has been unavoidably delayed. It will be given in our next number. . . . . . The Prize Problems are omitted in this number, in order to gain the month we have lost in the publication of Solutions. . . . . . We wish to call the particular attention of our readers to the chapters entitled "Notes and Queries;" and earnestly beg of all, to aid us in making this one of the most interesting and valuable departments of the MONTHLY. Under this head we shall be able to make good use of a large amount of material already on hand, without consuming as much space as would be required for giving the same matter in short articles. . . . . . In closing the first volume of the MONTHLY, we beg all, who have in any way aided us during the year, to accept our sincere thanks. Our duties, though not free from care, have yet been rendered pleasant by a considerate forbearance, not more for what has been done, than left undone. It is true that we have not been able to publish all the excellent matter we have received; but it is a satisfaction to know that the delay has not been misunderstood, and has caused us more anxiety than any of our contributors. The experiment of establishing a Mathematical Journal has been tried, and we point to Vol. I., as the most unmistakable evidence of success. While we acknowledge much valuable aid and counsel from friends in Cambridge and Boston, our special thanks are due to our friend and colleague, SIMON NEWCOMB, Esq., for the Indexes which accompany this number.

## LIST OF ERRATA.

On page 19, lines —1 and —2, for $\varrho_z$, read $\varrho_x$.

" 19, " —3 and —4, for $\varrho_z$, read $\varrho_y$.

" 21, line 10, for $1+(D_x y)^2$, read $(1+(D_x y)^2)^{\frac{1}{2}}$.

" 229, " —5, in the value of $y$, put exponent $\frac{1}{2}$ outside the parenthesis.

" 231, " 11, for "A D." read "that line."

" 258, Prob. II., for "Transpose," read "Transform."

" 263, line —3, for "$R$," read "$K$."

" 282, " —12, for "Art. 1135," read "Art. 1535."

" 338, last line, for "$> 0.250$," read "$< 0.250$."

" 346, line 11, for "multipliers," read "multiples."

" " last line, for $(s-1)$, read $(s+1)$.

" 347, line 10, read, one entering only $x$, another only $y$, &c.; another only $x$ and $y$, &c.; and one entering all $x,, z$.

" 348, Equation (11), for $Q^{P_3}$ read $a^{\lambda'}$, and insert

$$\lambda' = M(F\overline{\xi,\eta},\ F\overline{\xi,\zeta},\ F\overline{\eta,\zeta}) = F(M\overline{\xi,\eta},\ M\overline{\xi,\zeta},\ M\overline{\eta,\zeta}).$$

" 350, for 63 cents, read 36 cents.

" 360, $2r = d = 2R - a\pi \pm \sqrt{a^2\pi^2 + 4al - 8a\pi R}$, instead of the value there erroneously printed.

" 361, in the last value given of $d$, the parenthesis should be multiplied by $2a$, instead of 2.

" 373, Equation (2), take $\frac{1}{2}$ the radical.

" 374, line —4, for $x' = x,\ y = 1$, read $x' = 1$.

" 375, " —7, for "the value of $d$," read "two of the six values of $d$."

" 377, " —6, for $+px$, read $-px$.

" 378, line —3, for $-px$ read, $+px$.

" 379, Equation (28), for $\frac{1}{q}$ read $\frac{1}{9}$.

" 379, line —5, for "dropped," read "developed."

# ALPHABETICAL INDEX.

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